Information Acquisition and Revelation in the Financial Markets

A Dissertation

SUBMITTED TO THE FACULTY OF THE

UNIVERSITY OF MINNESOTA

BY

Yinxiao Chu

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Jan Werner, David Rahman

JUNE 2018

Acknowledgments

I would like to express my sincere appreciation and gratitude to my advisors, Prof. Jan Werner and Prof. David Rahman, for their patient guidance and mentorship he provided to me through to completion of this degree. Prof. Werner is exceptionally patient and offers critical suggestions. Prof. Rahman is very inspiring and thought-provoking. I am truly fortunate to have had the opportunity to work with them.

I would also like to thank my committee members, Prof. Aldo Rustichini and Prof. Hengjie Ai for the helpful discussion and suggestions. In a similar vein, I'd like to recognize the participants in Theory workshop.

Finally, I would like to thank my friends who support me for many years at the University of Minnesota.

For my dearest parents.

Abstract

Information plays an important role in financial markets. In this dissertation, first, we consider how traders choose different information. Second, we ask when traders acquire information under competition. Finally, we analyze how ambiguous information affects traders' incentives to trade and reveal their private information.

There is information not only about the payoff but also concerning the supply and demand of an asset. In Chapter 1, we study how traders choose to process different information while asset prices are conveying some information. We show that traders decide to process different types of information depends on their initial belief and the informativeness of asset prices. In particular, when the return to each type of information is increasing, traders choose to learn only one type of information. Those who have more precise initial belief about the asset payoff (supply) choose to learn more about the asset payoff (supply).

In Chapter 2, we study when traders decide to acquire information under competition. Traders consider two effects of competition in information acquisition: one is that an informed trader's profitability is affected by the presence of another informed trader, the other is the spillover of the information from the informed trader to the uninformed. We show that, when the former effect dominates, then traders tend to acquire information earlier. If the otherwise, then traders tend to delay their information acquisition.

In Chapter 3, we study traders' behavior when information is ambiguous, which gives rise to multiple probability models to describe uncertainty. We demonstrate that ambiguity will reduce traders' incentive to trade and reveal their private information. When there is a moderate level of ambiguity, informed traders start to trade randomly, whereas they trade for sure when there is no or a little uncertainty. When ambiguity is sufficiently large, informed traders choose not to trade any more, and no additional information will be revealed in the market.

Contents

List of Tables

List of Figures

Introduction

Financial markets are of great importance in the economy. People trade assets and commodities in various financial markets. There is a large amount of uncertainty involved in trading activities. Therefore, information comes into play. It helps traders to reduce uncertainty and to make sensible trading decisions. Different information possessed by traders also motivates them to trade with each other and to share their perceived risk.

There are numerous information sources. For example, to predict the movement of the S&P 500 index, investors can check the publicly available macroeconomic statistics and search for analysts' reports online. Speculators track investor sentiment in various media and statistics. Moreover, economists believe that traders form rational expectations. That is, asset prices reflect information about the fundamentals of the asset and of the economy; traders understand this relationship and can infer information from the asset prices.

The information is revealed through trading activities. Traders are not born with information. In the first place, they have to collect and process different items of information and then decide whether to engage in trading, so that information flows in financial markets. This dissertation aims to address the following issues regarding the role of information and traders' strategic interactions in the financial markets from different perspectives. First, we study how traders choose their information when they can infer some information through price. Second, we consider the timing of information acquisition when traders know that the trading activities of others reveal information. In addition, we investigate traders' behavior and how information revelation is affected when information is ambiguous. Ambiguity gives rise to multiple probability models to describe uncertainty.

Traders have to choose which information to process when they are faced with too much information and a limited capability to handle it. There are multiple sources of freely available information. We categorize the relevant information into two types: One is *payoff information* about the final payoff of the asset, e.g. financial reports, sales and revenue forecasts, analysts' reports. The other is *supply information* that signals the supply and demand from investors' liquidity motives, e.g. confidence and sentiment of investors, and exogenous liquidity shocks. However, there are limits and constraints on the human cognitive capability to process a large amount of information. The time and effort devoted to processing information are dubbed *attention*. A rational trader needs to decide how to allocate attention to different types of information.

Asset prices convey some information obtained by other traders. However, the price is also noisy, and it does not fully reveal the fundamental information. Therefore, in Chapter 1, we consider a model of an asset market where agents can choose their information. There are multiple rational expectation equilibria, which are solved explicitly, in an asset market where there are both private signals on payoff and supply information. Traders choose their private signals before the asset market opens. We point out the strategic interaction in attention allocation and provide sufficient conditions to guarantee the existence of equilibrium in general. There can be multiple overall equilibria in the game of attention allocation due to the multiplicity of equilibria in the asset market.

In particular, if information entropy measures information processing capacity, then agents allocate their attention to only one type of information because of increasing return to each type of information. We show that heterogeneity in the precisions of prior information can lead to a pattern of specialization. Traders with high initial prior precision about payoff information, i.e. with a comparative advantage in the payoff information, will choose to learn the payoff signals, while traders with low initial prior precision about payoff information have a comparative advantage in the supply information, and they will choose to learn the supply signals. Moreover, the payoff informativeness of the asset prices is higher (lower) in an equilibrium where fewer (more) agents are choosing to learn the payoff signals.

Then, we consider the time dimension of information acquisition and the fact that traders are not restricted to acquiring information at a specific time but can choose any time to obtain information. A trader's incentive can change over time. It depends on the evolution of market conditions and traders' prior knowledge enabling them to evaluate the benefit against the cost of acquiring information. For example, when the price of gold is rising, investors would like to investigate the potential profitability of gold-mining companies and decide whether to purchase their stocks.

When multiple traders become informed, the timing of their information acquisition is further complicated by competition. On the one hand, information reduces uncertainty and increases informed traders' expected profit. This is especially true when one trader acquires information earlier than the other. The first informed trader can enjoy an informational advantage and experience reduced competition in trading with other traders so that she can trade aggressively and earn more expected profits. On the other hand, while one trader pays the cost of acquiring information, some information can be disclosed to another trader who does not need to pay a penny. Uninformed traders can wait for free access to the information for which other traders have paid.

In Chapter 2, we examine an information acquisition timing game in the assets market. The market conditions, or the asset prices, fluctuate driven by public information and two traders decide when to acquire information on the asset payoff. Motivated by the trade-offs mentioned above, two specific externalities are present: *trading externality*, i.e. an informed trader's trading volume and profitability being affected by the presence of another informed trader, and *information externality*, i.e. the spillover of the information from the informed trader to the uninformed trader.

The former leads to the first-mover advantage or incentive for traders to acquire information earlier, and the latter gives rise to the second-mover advantage. Therefore, the nature of the timing game depends on the relative strength of the two externalities: if the first-mover advantage dominates, then the game of information acquisition timing is a preemption game; if the second-mover advantage dominates, then this game is a war of attrition. Consequently, compared to the timing decision on the information acquisition of only one trader, traders always acquire information later when the timing game is a war of attrition. However, whether or not traders acquire information earlier in a preemption game depends on the information externality.

Although traders can obtain a significant amount of information, the quality of the information is sometimes difficult to assess. We proceed to examine the effect of ambiguous information on trading behavior and information revelation. We say the information is *ambiguous* if there is no single probability distribution to describe the uncertainty. The situation of lacking knowledge of a unique probability distribution is reminiscent of the celebrated Ellsberg (1961) paradox. When agents choose between bets based on draws from an urn with a known distribution of balls of different colors and an urn with an unknown distribution, they exhibit *ambiguity aversion*, i.e. they are more likely to choose bets with known odds over the bets with unknown odds on the same stakes.

Ambiguity aversion can affect the incentives and the decisions of both traders and market makers because trading an asset is betting on its valuation. In Chapter 3, we study a sequential trading mechanism with ambiguity averse traders and market maker where the market maker posts bid and ask prices and traders, who may be informed or uninformed, arrive sequentially and decide whether or not to trade. Ambiguity can be a source of bid-ask spread in the sense that it increases if the belief in the market becomes more ambiguous.

In addition, there could be *informational cascades*, i.e. traders choose the same action sequentially regardless of their private information. This situation occurs when public beliefs are sufficiently ambiguous, due to the agents' aversion to uncertainty. Whenever there are informational cascades, prices fail to incorporate private information and the bid-ask spreads will not converge to the true value. Moreover, when private signals are ambiguous, trading activities that reveal the information imperfectly will inject ambiguity into the public beliefs. Therefore, the asset market will produce a situation that is arbitrarily close to an informational cascade even though it started out with little ambiguity at the beginning.

Unlike trading without ambiguity, where there is always some equilibrium with pure trading strategies, the trading game under ambiguity requires the informed traders to play mixed trading strategies to guarantee the existence of equilibrium. Because informed traders employ mixed trading strategies, and they may not trade, sometimes no-trade can also be informative about the true value of the asset. For example, this is the situation when signals are asymmetric: no news is bad (good) news in the market when a bad signal is less (more) informative than a good signal.

In short, we explore the role of information and traders' strategic interaction in the financial markets from different perspectives. The remaining chapters of the dissertation contain an in-depth discussion of these topics in the following order. First, traders choose their information in the sense that they allocate their attention to different types of information. Second, traders time their information acquisition when they are competing and some information is revealed in the activity of others. Finally, we see that ambiguity aversion affects the incentives for agents to trade and to reveal their private information in the asset market.

Chapter 1

Strategic Attention Allocation in the Asset Market

1.1 Introduction

Information plays an important role in the financial markets. There are numerous information sources, and plenty of information is freely available in the information era. For example, to predict the movement of S&P 500 index, investors can check the publicly available macroeconomic statistics and search for analyst reports online. Speculators trace the investor's sentiment from various media. Recently, some start-ups provide services to institutional investors based on their analysis of the "alternative data" such as social media data, online search trend, etc. Menkhoff (2010) analyzes survey data from 692 fund managers in five countries and finds that fund managers make their decision on fundamental analysis, technical analysis, and order flows.

We can categorize the relevant information into different types for a specific asset traded in the market. One type of the information—the payoff information–is about the fundamental value or final payoff of the asset. Examples of such information can be financial reports, sales and revenue forecasts, analyst reports, etc. Another type of information—the supply information–signals the excessive supply or demand from investors' liquidity motives. This type of information includes confidence and sentiment of investors, technical analysis of short-term price movements, exogenous liquidity shocks, and so forth, which is irrelevant to the fundamental value of the asset but affects investors' trading decision and the supply and demand of the asset.

Empirical studies reveal that asset prices and returns are affected by and reflect different information. The classical study of Fama and French (1992) shows that accounting information such as book-to-market ratio can predict cross-sectional returns. Non-fundamental information also has an influence on the asset prices and returns. Baker and Wurgler (2006) find that investor sentiment can explain some difference in the cross-sectional returns as well. Antweiler and Frank (2004) show that messages on the internet can predict the volatility of stock price.

Investors are not born with information but they have to collect and process different information. The information gathered and processed by the investors are manifested in their investment strategies and styles. Investors whose strategy based on fundamental analysis have to acquire payoff information such as accounting report, while those who are using technical analysis to formulate their investment strategy pay more attention to supply information such as the pattern of price movements. Evidence shows that investors employ different investment strategies and styles, which indirectly indicates that investors acquire different types of information. Farboodi and Veldkamp (2016) reveal that some hedge fund managers rely on fundamental analysis and others use quantitative strategies employing market transaction data.[1] Although

[1] Farboodi and Veldkamp (2016) use the Lipper TASS Database, which provides data on performance and other characteristics of 7,500 actively reporting hedge funds and 11,000 funds that are liquidated or stopped reporting. In their classification, a fund is "fundamental" if its strategy is explicitly based on fundamental analysis or upon the discretion of the manager; and a fund is "quantitative" if it deploys a technical and/or algorithmic strategy.

the importance of fundamental analysis is undeniable, Taylor and Allen (1992) report that over 90% of foreign exchange traders use some non-fundamental analysis.

There are limits and constraints of human cognitive capability to process a large amount of information, not to mention the pecuniary cost of acquiring information. With limited time and effort, investors can focus on collecting data only from some of the information sources, instead of collecting information from all available information. A rational trader needs to decide how to allocate her time and effort on different types of information. The time and effort devoted to processing information are dubbed *attention*.

On one hand, each trader is faced with too much available information but constrained with limited attention. On the other hand, rational traders should expect that asset prices convey some information obtained by other traders in the market equilibrium. But the price is also noisy, it is not fully revealing the fundamental information. As argued in Grossman and Stiglitz (1980), if the price is fully revealing all the information relevant to the asset fundamental, then traders will not have the incentive to acquire information. Larcker and Lys (1987) document that some risk arbitrageurs have superior information to predict the success of some events, such as mergers, tender offers, etc., and they earn a substantial excess return on their trading activities. Similar to that, attention is a scarce resource when the price is not fully revealing. Given that each trader anticipates information acquired by other traders reflected in publicly observable asset prices, how should a trader allocate her attention to acquire different types of information?

To explore this question, we consider a model of an asset market where agents can choose their information. First, traders choose to allocate their attention between two types of information, and then the asset market is open and they trade. Individual traders exhibit constant absolute risk aversion (CARA) and random variables are normally distributed. This CARA-normal rational expectation framework (see

Grossman and Stiglitz 1980; Verrecchia 1982; Admati 1985) is adopted to study the asset market where traders have differential information. Rational expectation equilibrium (REE) prices aggregate private information. In particular, the asset market model is built on that of Ganguli and Yang (2009), where additional private information on the net supply of the asset is allowed. Similar to their results, there could be multiple partially-revealing rational expectation equilibria (pr-REE) in the asset market. When agents have only private signals on the supply information, there can be an equilibrium where the asset price is purely noisy, not reflecting any payoff information.

In the first stage, traders choose to allocate attention among different types of information. To capture individual's limitation of information processing capacity, an information constraint is considered. Specifically, individuals choose how much attention they should allocate to reduce the uncertainty about payoff and supply information constrained by their information capacity.

The attention allocation problem in the first stage is strategic to the traders. Each trader anticipates that the asset price is reflecting what the "average" trader knows in an REE. However, such average is determined by the profile of individual strategies to allocate attention. Therefore, this formulation makes the first-stage problem an aggregative game, where their expected utility depends only on their own action and the aggregation of all individual choices. Some authors, e.g. Schmeidler (1973), Mas-Colell (1984), and Rath (1992), have shown that there exist pure strategy Nash equilibria in this type of games. However, the game of attention allocation violates some the assumptions of a standard aggregative game, therefore the existence of equilibrium requires additional conditions. We give some sufficient condition to guarantee the existence of equilibrium in general. There can be multiple overall equilibria in the game of attention allocation due to the multiplicity of equilibria in the asset market.

If private signals are exogenously given, the asset price is purely noisy in one of the possible equilibria when agents only have signals on the supply information. However, when agents allocate their attention, the asset price cannot be purely noisy even though it is the case that all agents choose to only process the supply information.

We further consider a specific information processing capacity measure, information entropy, introduced by Sims (2003) in the economic literature from information theory (e.g. Shannon 1948; Khinchin 2013). Since information has increasing returns to its scale under the information entropy, traders specialize, learning either the payoff signals or the supply signals only. There can exist equilibrium where all the traders choose to learn the same type information.

If traders are different and some have comparative advantages in different types of information; specifically, they are heterogeneous in their prior precisions of payoff information, then a pattern of specialization may emerge in equilibrium. Traders with high initial prior precision about payoff information, i.e. with the comparative advantage in the payoff information, will choose to learn the payoff signals, while traders with low initial prior precision about payoff information have comparative advantage in the supply information, and they will choose to learn the supply signals.

Some empirical facts suggestively support our result. The data in Farboodi and Veldkamp (2016) shows that investment styles of the hedge funds are typically specialized: half of the asset is managed by funds that specialize in either fundamental analysis or quantitative analysis over the entire period from 1994 to 2016. Menkhoff (2010) documents that smaller funds rely on technical analysis more heavily than larger funds, which reflects the differentials in capability or cost for acquiring fundamental information as he argues. If we look at the statistics on retail investors who are considered disadvantageous in the fundamental information, Hoffmann and Shefrin (2014) reveal that only 20% of them uses fundamental analysis.[2]

[2] Hoffmann and Shefrin (2014) use a data set from a discount online broker in the Netherlands from 2000 to 2006.

If there exist two equilibria in the same economy, then they have different comparative statics, changing in the different directions with respect to the same change in the parameter. In one equilibrium, the payoff informativeness of the price is increasing in the agent's information capacity and prior precisions but decreasing in the risk aversion. However, in another equilibrium, the payoff informativeness of the price changes in the opposite direction with respect to the same change in the parameters.

Related Literature

Ganguli and Yang (2009) first consider payoff and supply signals in an REE framework. They show that there are multiple REE with partially revealing prices. The asset market model in this paper closely follows theirs. They consider agents with identical preference and two types, informed and uninformed, and study the costly information acquisition with fixed signal precisions in such environment. We extend their model where agents are allowed to have different preferences and arbitrary precisions of signals so that we can analyze the problem of how agents allocate attention and choose signal precisions.

There are some studies on the entropy-type information constraint and attention allocation in an REE framework. For example, Van Nieuwerburgh and Veldkamp (2009) and Mondria (2010) consider a multi-asset environment and study how agents allocate their attention to the fundamental information of different assets. In particular, Van Nieuwerburgh and Veldkamp (2009) explain that home investors specialize to learn about home asset and invest more heavily on home asset because they have an initial information advantage, a higher prior precision of returns, on the home asset. Similarly, in our model, traders with higher initial precision about fundamental value will choose to learn the fundamental signals, although our paper studies how agents allocate attention to different types of information, fundamental and non-fundamental.

The model of Farboodi and Veldkamp (2016) is the closest to ours in spirit. They also consider attention allocation to different types of information, though, in an overlapping generation version of REE. They want to explain the the increasing information capacity lead to the switching of hedge fund styles from fundamental analysis to quantitative analysis. For this purpose, they assume a specific liquidity shock so that there is a unique equilibrium and they focus on a quadratic information constraint. Despite the difference in modeling choices, one similar result between their study and ours stands out: the individual problem of maximizing the expected utility is equivalent to maximize the posterior precision of the asset payoff.

The rest of the paper is organized as follow. Section 1.2 describes the model of an asset market where there are both payoff and supply signals. We then consider the equilibrium of attention allocation in Section 1.3. Section 1.4 analyzes the the model in which the information capacity is measured by information entropy. Section 1.5 discuss the existence of equilibrium in the general model. Section 1.6 concludes and discusses implications of the model. All the proofs are in the Appendix to Chapter 1.

1.2 The Asset Market with Payoff and Supply Signals

We begin with the description of the asset market with heterogeneous agents and arbitrary parameters of the signal structure. It extends the asset market model of Ganguli and Yang (2009) with payoff and supply signals. There is a risk-free asset, a risky asset, and a continuum of agents with measure one.[3] The asset market model is static: The agents observe their private signals, choose portfolios, and then they consume the final wealth.

[3] We think of a unit interval of agents, and the set of agents is naturally endowed with a Lebesgue measure.

Endowment and Preference

The agents, denoted by $a \in [0,1]$, are endowed with the same initial wealth W. The initial wealth can be considered as holding risk-free bond with gross return R, which serves as the numeraire in the economy. The payoff $\tilde{\theta}$ of the risky asset is random. Let q_a be the quantity of the risky asset held by an agent a and $\tilde{p}$ be the price of the risky asset. The final wealth of the agent is

$$WR + q_a(\tilde{\theta} - \tilde{p}R). \tag{1.2.1}$$

The agent's objective is to maximize the expected utility of the final wealth. Each agent has a von Neumann-Morgenstern (vNM) utility function with constant absolute risk aversion (CARA) and their risk aversion λ_a may differ from each other. That is, the vNM utility function for agent a has the form

$$u_a(w) = -e^{-\lambda_a w}. \tag{1.2.2}$$

Prior Information on the Payoff and Supply

Each agent $a \in [0,1]$ has a prior belief that the payoff $\tilde{\theta}$ follows the normal distribution $N(\bar{\theta}, \Theta_a)$. In the standard rational expectation framework, agents are assumed to have the same prior. However, with some departure from the standard common prior assumption, agents are also allowed to differ in the variance Θ_a of their prior belief about the asset payoff. Agents may differ in confidence, experience, or expertise. For example, if agents use different estimators, simple or sophisticated, then they will have different standard errors even though agents are faced with the same data *a priori*. Kandel and Pearson (1995) provide empirical evidence to support that traders interpret public data differently.

The net supply $\tilde{\zeta}$ of the risky asset is also random; such randomness is due to the existence of liquidity traders who trade the asset for exogenous reasons. The noisy supply $\tilde{\zeta}$ is also normally distributed and agents share the same mean $\bar{\zeta}$ but may differ in the variance Z_a in their prior beliefs, i.e. $\tilde{\zeta} \sim N(\bar{\zeta}, \mathrm{Z}_a)$ for agent $a \in [0,1]$. We often use on the *precisions*, the inverse of variances, of the prior information about payoff and supply, i.e. $\pi_{\theta,a} = \Theta_a^{-1}$ and $\pi_{\zeta,a} = \mathrm{Z}_a^{-1}$, which turns out more convenient to work with in the calculation.

The Payoff and Supply Signals

Each agent a can observe two private signals, $\tilde{s}_a$ and $\tilde{z}_a$, on the asset payoff and noisy supply, respectively. Specifically, the signals that each agent observes when asset market opens are assumed to be of the following form,

$$\tilde{s}_a = \tilde{\theta} + \tilde{\epsilon}_a,\ \tilde{\epsilon}_a \sim N\left(0, \mathcal{E}_a\right), \tag{1.2.3}$$

$$\tilde{z}_a = \tilde{\zeta} + \tilde{\xi}_a,\ \tilde{\xi}_a \sim N\left(0, \mathcal{X}_a\right), \tag{1.2.4}$$

where $\left(\tilde{\theta}, \tilde{\zeta}, (\tilde{\epsilon}_a, \tilde{\xi}_a)_{a\in[0,1]}\right)$ are mutually independent. Notice that the private signals have the same structure among agents, however, the variances, $\mathcal{E}_a$ and $\mathcal{X}_a$, of the noise terms, $\tilde{\epsilon}_a$ and $\tilde{\xi}_a$, can vary by agents. In this section, the signal structure is given and differences in signal variances are due to the fact the individual agents have diversified information sources. When we analyze the attention allocation problem in Section 1.3, agents can choose these parameters, as collecting their own information, to determine their signal structures. The precisions of signals are $\sigma_{\theta,a} = \mathcal{E}_a^{-1}$ and $\sigma_{\zeta,a} = \mathcal{X}_a^{-1}$.

We assume that the *Strong Law of Large Numbers (SLLN)* hold for this economy with a continuum of agents by convention. Let $\{\nu_a\}_{a\in[0,1]}$ be a process of independent random variables with $E[\nu_a] = 0$ for all a and the variance $Var[\nu_a]$ are uniformly bounded. Then $\int_0^1 \nu_a da = 0$ almost surely. [4]

The economy can be described as a measurable function $(\lambda_\cdot, \pi_{\theta,\cdot}, \pi_{\zeta,\cdot}, \sigma_{\theta,\cdot}, \sigma_{\zeta,\cdot}) : [0,1] \to \mathbb{R}^5_+$, where λ_a, $\pi_{\theta,a}$, $\pi_{\zeta,a}$, $\sigma_{\theta,a}$, and $\sigma_{\zeta,a}$ are the risk aversion, prior precisions of the payoff and supply information, precisions of the payoff and supply signals, respectively, of agent a. For a complete description of priors and signals, we follow conventions when precisions of prior beliefs or signals are equal to zero: If either $\pi_{\theta,a} = 0$ or $\pi_{\zeta,a} = 0$, then the agent a has an improper prior of either payoff or supply information; if either $\sigma_{\theta,a} = 0$ or $\sigma_{\zeta,a} = 0$, then the agent a is not observing either a payoff signal or a supply signal.

1.2.1 Rational Expectation Equilibrium

As argued in the earlier literature, e.g. Hellwig (1980), because individual agent's demand for asset reflects her private information, the asset price that clears the market has to be a function of private information $(s_a, z_a)_{a\in[0,1]}$ of all agents. The asset price typically provides additional information to each agent beyond their own information. The agents know the price function and understand the actual joint distribution of private signals and the underlying aggregate random variables in equilibrium. Therefore, each agent a exploits the information contained in the asset price and takes expectation conditional on her private signals, s_a and z_a, and the asset price $\tilde{p}$. Her expected utility is

$$E_a\left[u_a\left[WR + q_a(\tilde{\theta} - \tilde{p}R)\right] |\tilde{s}_a, \tilde{z}_a, \tilde{p}\right]. \tag{1.2.5}$$

[4] Doob (1937) and Judd (1985) point out the issue of measurability of the process $(\tilde{x}_a)_{a\in[0,1]}$. Other authors (e.g. Admati, 1985; Vives, 2010) argue potential resolutions to this technical issue. Think that the integral is approximated by the average of discrete random variables and the average converges by the usually strong law of large number. The integral is then defined as a limit when number of discrete random variables increases. See also Sun (2006).

Hellwig (1980) considers an asset market with finite numbers of agents where the equilibrium price linearly depends on all the private signals. The price has two roles: one is to clear the market, and the other is to convey information of all other agents. The asset demand of each agent depends on her own private signals and market clearing equation make the price an implicit function of all agents' private signals. From each agent's perspective, the price contains information beyond her own signals and rational agents understand the joint distribution of private signals and price. Hellwig also argues that, if the number of agents grows to infinity, the price converges to a function that only depends on the aggregate shocks, namely, $\tilde{\theta}$ and $\tilde{\zeta}$. In a large economy, Admati (1985) shows the existence of an equilibrium price function that depends on the aggregate shocks where idiosyncratic shocks are aggregated out by SLLN.

Denote $q_a(s_a, z_a, p)$ the asset demand function of agent a who observes signals s_a and z_a and price p, resulting from maximization of her expected utility (2.2.4). Then the following definition of an equilibrium in the asset market is standard.

Definition 1.2.1. A *rational expectation equilibrium (REE)* in the asset market consists of an asset price $\tilde{p}$ and asset demand functions $\{q_a(\tilde{s}_a, \tilde{z}_a, \tilde{p})\}_{a\in[0,1]}$ such that:

(i) $\tilde{p}$ is $(\tilde{\theta}, \tilde{\zeta})$ measurable,

(ii) portfolios are optimally chosen

$$q_a(\tilde{s}_a, \tilde{z}_a, \tilde{p}) \in \arg\max_q E_a \left[u_a \left[WR + q(\tilde{\theta} - \tilde{p}R)\right] |\tilde{s}_a, \tilde{z}_a, \tilde{p}\right], \tag{1.2.6}$$

and

(iii) asset market clears, i.e. equation

$$\int_0^1 q_a(\tilde{s}_a, \tilde{z}_a, \tilde{p})da = \tilde{\zeta} \tag{1.2.7}$$

holds almost surely.

The Partially-Revealing Rational Expectation Equilibrium

We consider a class of REE with linear *partially-revealing* (*pr-REE*) price function of the following form:

$$\tilde{p} = c\left(x + D\tilde{\theta} - \tilde{\zeta}\right) \tag{1.2.8}$$

with $c > 0$ and $D \geq 0$.[5] The information of the payoff value $\tilde{\theta}$ is revealed in the price function when $D > 0$, though it is also confounded by the noisy supply $\tilde{\zeta}$. Since the coefficient of $\tilde{\zeta}$ is normalized to unit, the coefficient D of $\tilde{\theta}$ is considered as the (relative) *payoff informativeness* of the price function. Larger D indicates that price is more responsive to the change in the payoff information. As a special case, the REE can be *purely noisy* if $D = 0$ and the price is not revealing any information about the payoff of the asset.

Before proceeding further, boundedness assumptions are imposed so that the key integrals will be well-defined. First, assume that there exist two numbers β and γ such that $0 < \beta < \gamma < \infty$ and $\beta \leq \lambda_a \leq \gamma$ for almost every $a \in [0,1]$. Second, $\pi_{\theta,a}$, $\pi_{\zeta,a}$, $\sigma_{\theta,a}$, and $\sigma_{\zeta,a}$ are uniformly bounded. Then some key integrals, especially,

$$\hat{\Sigma}_\theta = \int_0^1 \lambda_a^{-1}\sigma_{\theta,a}da, \tag{1.2.9}$$

$$\hat{\Sigma}_\zeta = \int_0^1 \lambda_a^{-1}\sigma_{\zeta,a}da, \tag{1.2.10}$$

are well-defined. Moreover, for a strictly positive measure of agents, they have either $\sigma_{\theta,a} > 0$ or $\sigma_{\zeta,a} > 0$ so that at least one of the aggregates $\hat{\Sigma}_\theta$ and $\hat{\Sigma}_\zeta$ must be strictly positive. Let $\hat{\Pi}_\theta = \int_0^1 \lambda_a^{-1}\pi_{\theta,a}da$ and $\hat{\Pi}_\zeta = \int_0^1 \lambda_a^{-1}\pi_{\zeta,a}da$. The next proposition shows how these aggregates determine the existence of equilibria in the asset market.

[5] We can consider first the price function has the form $\tilde{p} = \frac{1}{R}\left(A + B\tilde{\theta} - C\tilde{\zeta}\right)$ with $B > 0$ and $C > 0$. The price is revealing the payoff information because $B > 0$, but it is only partially revealing due to that $C > 0$. Then, we can get the equivalent expression (1.2.8) of price function by letting $c = C/R$, $x = A/C$, and $D = B/C$.

Theorem 1.2.1. *If $0 < \hat{\Sigma}_\theta \hat{\Sigma}_\zeta < 1/4$, then there exist two pr-REE with the asset price function $\tilde{p} = c\left(x + D\tilde{\theta} - \tilde{\zeta}\right)$ where D solves the quadratic equation*

$$\hat{\Sigma}_\zeta D^2 - D + \hat{\Sigma}_\theta = 0, \tag{1.2.11}$$

and

$$c = \frac{D\hat{\Pi}_\zeta + 1}{R\left[\hat{\Pi}_\theta + \hat{\Sigma}_\theta + D^2\left(\hat{\Pi}_\zeta + \hat{\Sigma}_\zeta\right)\right]}, \tag{1.2.12}$$

$$x = \frac{\hat{\Pi}_\theta \bar{\theta} + D\hat{\Pi}_\zeta \bar{\zeta}}{D\hat{\Pi}_\zeta + 1}. \tag{1.2.13}$$

Remark 1.2.1. When $\hat{\Sigma}_\theta \hat{\Sigma}_\zeta = 1/4$, there exists only one pr-REE with $D = 1/2\hat{\Sigma}_\zeta$. When $\hat{\Sigma}_\zeta = 0$, there exists one pr-REE with $D = \hat{\Sigma}_\theta$. When $\hat{\Sigma}_\theta = 0$, there exist two pr-REE with $D = 0$ or $D = 1/\hat{\Sigma}_\zeta$.

$\hat{\Sigma}_\theta$ and $\hat{\Sigma}_\zeta$ are the average of signal precisions weighted by risk tolerance, i.e. the inverse of risk aversion. Such a pr-REE exists only when

$$\hat{\Sigma}_\theta \hat{\Sigma}_\zeta \leq 1/4. \tag{1.2.14}$$

This is equivalent to a joint restriction on the magnitude of two aggregates. It requires that the average signal precisions small enough, i.e. the signals are noisy enough on average, or agents are sufficiently risk averse to prevent agents to trade too much to reveal a large amount of information.

Note that, when $0 < \hat{\Sigma}_\theta \hat{\Sigma}_\zeta < 1/4$, the quadratic equation (1.2.11) has two positive roots. We distinguish two different roots by denoting a COM equilibrium if

$$D^{COM} = \frac{1 + \sqrt{1 - 4\hat{\Sigma}_\theta \hat{\Sigma}_\zeta}}{2\hat{\Sigma}_\zeta}, \tag{1.2.15}$$

and denote a GS equilibrium if

$$D^{GS} = \frac{1 - \sqrt{1 - 4\hat{\Sigma}_\theta \hat{\Sigma}_\zeta}}{2\hat{\Sigma}_\zeta}. \tag{1.2.16}$$

Recall the Equation (1.2.9), $\hat{\Sigma}_\theta$ is the average of precisions of payoff signals weighted by the risk tolerance, so it can be considered as a measurement of the average payoff information in the market. D^{GS} is increasing in $\hat{\Sigma}_\theta$, which means that the payoff informativeness of the price increases as the average payoff information increases in a GS equilibrium. Moreover, $D^{GS} = 0$ when $\hat{\Sigma}_\theta = 0$. The GS equilibrium reduces to a pn-REE when no individual has any payoff signals. Note that when $\hat{\Sigma}_\zeta = 0$, the Equation (1.2.11) is reduced to a linear equation and there is a unique positive solution $\hat{\Sigma}_\theta$ and it is a GS equilibrium since $\lim_{\hat{\Sigma}_\zeta \to 0} D^{GS} = \hat{\Sigma}_\theta$.

The other equilibrium D^{COM} has different properties. This equilibrium is novel in the rational expectation framework, and it appears only when there are supply signals in the market. D^{COM} is decreasing in $\hat{\Sigma}_\theta$, which indicates that the payoff informativeness of the price decreases as the average payoff information increases in a COM equilibrium. When $\hat{\Sigma}_\theta = 0$, the only pr-REE is $D^{COM} = 1/\hat{\Sigma}_\zeta$. When $\hat{\Sigma}_\zeta \to 0$, $D^{COM} \to \infty$, i.e. D^{COM} becomes fully-revealing if there is no private signals of supply information.

Ganguli and Yang (2009) study the information acquisition problem in an asset market with both payoff and supply signals. They show that information acquisition exhibits strategic substitutability, which is in line with the results of Grossman and Stiglitz (1980), if agents all anticipate a GS equilibrium. But information acquisition exhibits strategic complementarity if agents coordinate on a COM equilibrium, therefore there can be multiple equilibria in the information market if all agent anticipate a COM equilibrium.

Once a value of D is solved from Equation (1.2.11), the values of c and x can also be determined along with the given aggregates. Therefore, D itself characterizes a pr-REE.

Mathematically, the multiplicity of pr-REE is a consequence of $\hat{\Sigma}_\zeta > 0$ from equation (1.2.11). That is, private signals on the supply information is crucial for the multiplicity of equilibria. Intuitively, when agents have private signals on the supply information, they can coordinate their demands through the aggregate net supply of the asset. If they conjecture that the price has a relatively high (low) payoff-informativeness, then their demands are more (less) sensitive to the payoff signals. Then the aggregate demand is more (less) sensitive to the payoff information and hence price self-fulfills to be high (low) responsive to the payoff information. See more details in Ganguli and Yang (2009).

Notice that there exists a pn-REE when the aggregation of private payoff information is zero. The existence of pn-REE is also irritating because it states that the asset price can be irrelevant to its payoff in the rational expectation and agents can learn nothing from the asset price. If this is the case, agents will have stronger incentive to acquire private information about the payoff. Diamond and Verrecchia (1981) and Admati (1985) argue that the asset price can be purely noisy as a limiting case when the prior variance of the supply information tends to infinity, while Lintner (1969) has a purely noisy asset price in equilibrium by explicitly assuming that agents do not use the information from price. However, the existence of pn-REE here is under the conditions with finite prior variance of the supply information and agents forms rational expectations.

1.3 The Equilibrium with Attention Allocation

The signal structure was fixed in Section 1.2. Now consider the problem where the agents have the ability to choose to learn the private signals in the sense that they can determine how precise their private signals are. Before the asset market opens, agents choose how much information to learn. They collect data from different information sources, classify them into two types information regarding payoff $\tilde{\theta}$ and supply $\tilde{\zeta}$. Then each agent a synthesize the information she collects into two unbiased signals $\tilde{s}_a$ and $\tilde{z}_a$ of two types of information. Because of the normality assumptions, agents essentially choose the precisions of two signals. The more data collected on the payoff information and/or the supply information, the higher precisions $\sigma_{\theta,a}$ and/or $\sigma_{\zeta,a}$ of the payoff signal and/or the supply signal would be. Although gathering data and improving the precisions of the signals are free of monetary charge, it requires to allocate *attention*, the agents' time and efforts to process information.

1.3.1 The Information Constraint and the Action Space

Since the signals are normally distributed, the information constraint of agent a can be consider as a nonempty and compact subset $K_a \subseteq \mathbb{R}_+^2$ of signal precisions $(\sigma_{\theta,a}, \sigma_{\zeta,a})$.[6] For example, following Sims (2003), the amount of information contained in the signals can be quantified by using the concepts from information theory. Specifically, *information entropy* (Shannon, 1948) is used in the information theory to measure the unpredictability of a random vector or the average amount of information generated by data drawn from a distribution, and it can be derived from some reasonable axioms (e.g. Khinchin, 2013). In general, let $\tilde{\omega}$ be a n-dimensional random vector on some probability space with a continuous density $p(\omega)$. The information entropy $H(\tilde{\omega})$ is

[6] Assume that $(0, 0) \in K_a$ and it is non-singleton.

defined by

$$H(\tilde{\omega}) = -E\left[\ln p(\omega)\right] = -\int p(\omega)\ln p(\omega)d\omega. \tag{1.3.1}$$

For example, for a normally distributed random vector $\tilde{\omega} \sim N(\mu, \Sigma)$, the entropy is

$$H(\tilde{\omega}) = -\frac{1}{2}\ln\left[(2\pi e)^n|\Sigma|\right], \tag{1.3.2}$$

where $|\Sigma|$ is the determinant of Σ. Intuitively, the larger the variance, the more unpredictable is the random vector.

Consider two random vectors $\tilde{\omega}$ and $\tilde{\nu}$ having a joint density function $p(\omega, \nu)$ with marginal densities $p(\omega)$ and $p(\nu)$. Their *mutual information* $I(\tilde{\omega}; \tilde{\nu})$ measures the amount of information contained in one of the random vectors through another. The mutual information $I(\tilde{\omega}; \tilde{\nu})$ is defined as

$$I(\tilde{\omega}; \tilde{\nu}) = H(\tilde{\omega}) - H(\tilde{\omega}|\tilde{\nu}) = \int\int p(\omega, \nu)\ln\frac{p(\omega, \nu)}{p(\omega)p(\nu)}d\omega d\nu. \tag{1.3.3}$$

Intuitively, mutual information measures the amount of information that $\tilde{\omega}$ and $\tilde{\nu}$ share: the amount of uncertainty of one random vector reduced by knowing the other. In one extreme case, if $\tilde{\omega}$ and $\tilde{\nu}$ are independent, then $\tilde{\omega}$ and $\tilde{\omega}|\tilde{\nu}$ have the same distribution, and knowing the value of $\tilde{\nu}$ is not helpful for reducing uncertainty of $\tilde{\omega}$ at all, and hence their mutual information is zero. In the other extreme case, if two random vectors are perfectly correlated, then knowing $\tilde{\nu}$ eliminates all the uncertainty about $\tilde{\omega}$, therefore, in this case, the mutual information is the information entropy of $\tilde{\omega}$. In addition, one can show that mutual information is symmetric, i.e. $I(\tilde{\omega}; \tilde{\nu}) = I(\tilde{\nu}; \tilde{\omega})$.

Each agent is endowed with $h_a > 0$ units of attention resources. The *information capacity constraint* restricts the amount of information about $(\tilde{\theta}, \tilde{\zeta})$ contained in the signals $(\tilde{s}_a, \tilde{z}_a)$, that is, the mutual information $I(\tilde{\theta}, \tilde{\zeta}; \tilde{s}_a, \tilde{z}_a)$ of $(\tilde{\theta}, \tilde{\zeta})$ and $(\tilde{s}_a, \tilde{z}_a)$, not exceeding individual agent's attention resources, i.e.

$$I(\tilde{\theta}, \tilde{\zeta}; \tilde{s}_a, \tilde{z}_a) \leq h_a. \tag{1.3.4}$$

Because $(\tilde{\theta}, \tilde{\zeta})$ and $(\tilde{\theta}, \tilde{\zeta})|(\tilde{s}_a, \tilde{z}_a)$ are normally distributed, $I(\tilde{\theta}, \tilde{\zeta}; \tilde{s}_a, \tilde{z}_a) = \frac{1}{2}\ln\left[\frac{\pi_{\theta,a}+\sigma_{\theta,a}}{\pi_{\theta,a}}\frac{\pi_{\zeta,a}+\sigma_{\zeta,a}}{\pi_{\zeta,a}}\right]$ by applying Equation (1.3.2). The information constraint (1.3.4) is equivalent to

$$\frac{\pi_{\theta,a} + \sigma_{\theta,a}}{\pi_{\theta,a}} \frac{\pi_{\zeta,a} + \sigma_{\zeta,a}}{\pi_{\zeta,a}} \leq H_a \tag{1.3.5}$$

where $H_a = e^{2h_a}$. We refer H_a as the measurement of *information capacity*, and define the action space K_a of individual agent $a \in [0, 1]$ by

$$K_a = \left\{ \sigma_{\theta,a}, \sigma_{\zeta,a} \geq 0 \,\middle|\, \frac{\pi_{\theta,a} + \sigma_{\theta,a}}{\pi_{\theta,a}} \frac{\pi_{\zeta,a} + \sigma_{\zeta,a}}{\pi_{\zeta,a}} \leq H_a \right\}. \tag{1.3.6}$$

It is evident that K_a is a compact subset in $\mathbb{R}^2_+$.

Although Sims (2003) introduces the information constraint from the information theory literature, in principle there could be other choices of information constraint. One type of constraint considered by Van Nieuwerburgh and Veldkamp (2010) is an additive constraint.

$$\rho_{\epsilon,a} + \rho_{\xi,a} \leq H_a. \tag{1.3.7}$$

The interpretation for this constraint is that the learning technology is analogous to a sequence of independent draws of either a payoff or a supply signal with precision δ. Each independent draw of a normally distributed signal adds precision δ to the posterior belief. Constraining the sum of incremental precisions of the posterior belief not exceeding H_a is equivalent to restrict the total number of draws on payoff and supply signals to be $N \leq H_a/\delta$.

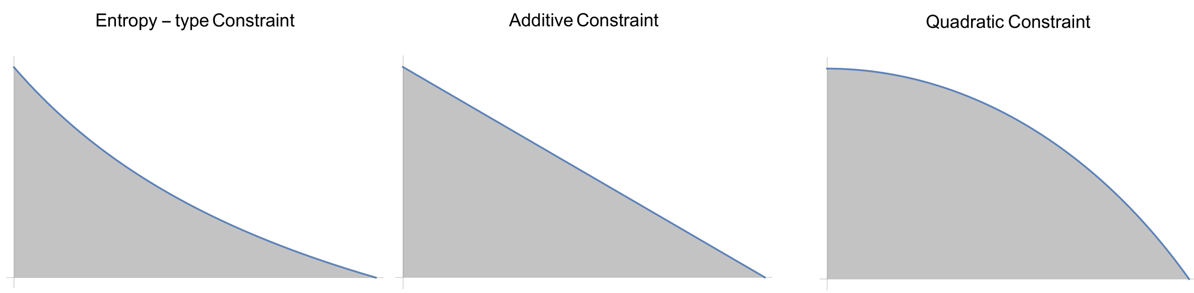

Figure 1.3.1: Examples of Information Constraints

Another possible information constraint studied by Farboodi and Veldkamp (2016) is a quadratic constraint,

$$\rho_{\epsilon,a}^2 + \chi\left(\rho_{\zeta,a} + \rho_{\xi,a}\right)^2 \leq H_a. \tag{1.3.8}$$

This constraint captures idea that it is getting tougher and tougher to acquire more and more precise information about a given random variable, while the total cost of acquiring two different types of information is additive.

For all these popular choice of information constraints, they are essentially restricting the magnitude of signal precisions $\rho_{\epsilon,a}$ and $\rho_{\xi,a}$ to determine the signal structure of each agent. Therefore, in general, each agent's action space can be represented as a constraint set of choice of signal precisions $\rho_{\epsilon,a}$ and $\rho_{\xi,a}$, which is also a compact subset $K_a \subseteq \mathbb{R}_+^2$.

The shaded area in the Figure 1.3.1 depicts some examples of the three types of information constraint sets discussed above. Then main distinctions among them is the convexity of the constraint sets. The entropy-type constraint has a convex complement, while the quadratic constraint set itself is convex. Though the solutions to constraint optimization problems also depends on the objective function, the entropy-type constraint more likely admits corner solutions. And this is exactly the case when later we solve the attention allocation problem where the objective function is effectively linear.

1.3.2 Strategic Attention Allocation

When the agents allocate their attention, the signals have not been realized. Therefore, each agent's expectation is taken over the prior information in the stage of choosing precisions of signals, i.e. the indirect utility of allocation attention to signal $(\tilde{s}_a, \tilde{z}_a)$ is

$$E_a \left[-\exp \left\{ -\lambda \left[WR + q_a(\tilde{\theta} - \tilde{p}R) \right] \right\} \right] \tag{1.3.9}$$

where $\tilde{p}$ be a pr-REE equilibrium price function, and q_a is the portfolio choice of agent a.

Applying the law of iterated expectation to Equation (1.3.9) and after some algebra of exponential functions and normal distributions, the indirect utility for such agent a who takes $\hat{\Sigma}_\theta$ and $\hat{\Sigma}_\zeta$ as given, expects a pr-REE D, and chooses her information strategy $\sigma_{\theta,a}$ and $\sigma_{\zeta,a}$ is

$$U_a^D \left(\sigma_{\theta,a}, \sigma_{\zeta,a}; \hat{\Sigma}_\theta, \hat{\Sigma}_\zeta \right) = E_a \left[-\exp \left[-\lambda_a WR - \frac{\pi'_a}{2} \left(\theta'_a - \tilde{p}R \right)^2 \right] \right], \tag{1.3.10}$$

where $\pi'_a = Var_a \left[\tilde{\theta} | \tilde{s}_a, \tilde{z}_a, \tilde{p} \right]$ and $\theta'_a = E_a \left[\tilde{\theta} | \tilde{s}_a, \tilde{z}_a, \tilde{p} \right]$ are the posterior variance and mean, respectively, of agent a.[7] The choice of individual precision $(\sigma_{\theta,a}, \sigma_{\zeta,a})$ of signals affects the posterior mean and variance of individual's estimates. Moreover, the profile of attention allocation affects the price and individual estimates through the aggregates $\hat{\Sigma}_\theta$ and $\hat{\Sigma}_\zeta$. Hence, the attention allocation is strategic, meaning that individual expected utility depends on the choice of others. To emphasize such dependence of the expected utility on $(\sigma_{\theta,a}, \sigma_{\zeta,a})$ and $\left(\hat{\Sigma}_\theta, \hat{\Sigma}_\zeta \right)$, we denote this indi-

[7]Explicitly,

$$\begin{aligned} \pi'_a &= \pi_{\theta,a} + \sigma_{\theta,a} + D^2 \left(\pi_{\zeta,a} + \sigma_{\zeta,a} \right), \\ \theta'_a &= \left(\pi'_a \right)^{-1} \Big\{ \pi_{\theta,a} \bar{\theta} + D \pi_{\zeta,a} \bar{\zeta} + \left[\sigma_{\theta,a} + D^2 \left(\pi_{\zeta,a} + \sigma_{\zeta,a} \right) \right] \tilde{\theta} \\ &\qquad - D \pi_{\zeta,a} \tilde{\zeta} + \sigma_{\theta,a} \tilde{\epsilon}_a + D \sigma_{\zeta,a} \tilde{\xi}_a \Big\}. \end{aligned}$$

rect utility $U_a^D\left(\sigma_{\theta,a},\sigma_{\zeta,a};\hat{\Sigma}_\theta,\hat{\Sigma}_\zeta\right)$, *the expected utility corresponding to asset market equilibrium D.*

Linear Objective Function

The individual problem of attention allocation is

$$\max_{\left(\sigma_{\theta,a},\sigma_{\zeta,a}\right)\in K_a} U_a^D\left(\sigma_{\theta,a},\sigma_{\zeta,a};\hat{\Sigma}_\theta,\hat{\Sigma}_\zeta\right). \tag{1.3.11}$$

In the following proposition, it shows that U_a^D depends on $(\sigma_{\theta,a},\sigma_{\zeta,a})$ only through the posterior precision π'_a, where

$$\pi'_a=\pi_{\theta,a}+\sigma_{\theta,a}+D^2\left(\pi_{\zeta,a}+\sigma_{\zeta,a}\right), \tag{1.3.12}$$

and U_a^D is strictly increasing in the posterior precision π'_a. Hence, the individual problem can be simplified to allocate attention so as to maximize her posterior precision.

Proposition 1.3.1. *Given the aggregates* $\left(\hat{\Sigma}_\theta,\hat{\Sigma}_\zeta\right)$ *and an asset market equilibrium D, the individual problem of attention allocation (1.3.11) is equivalent to maximize the posterior precision (1.3.12) subject to the information constraint* $(\sigma_{\theta,a},\sigma_{\zeta,a})\in K_a$.

Equilibrium

A *Nash equilibrium* in the game of attention allocation is a strategy profile $(\sigma_{\theta,a},\sigma_{\zeta,a})_{a\in[0,1]}$ such that

$$U_a^D\left(\sigma_{\theta,a},\sigma_{\zeta,a};\hat{\Sigma}_\theta,\hat{\Sigma}_\zeta\right)\geq U_a^D\left(\sigma_\theta,\sigma_\zeta;\hat{\Sigma}_\theta,\hat{\Sigma}_\zeta\right), \tag{1.3.13}$$

for all $(\sigma_\theta,\sigma_\zeta)\in K_a$ and almost every $a\in[0,1]$.

The standard definition of an *overall equilibrium* of this game of attention allocation is as follow: an equilibrium consists of a profile of attention allocation $(\sigma_{\theta,a}, \sigma_{\zeta,a})_{a\in[0,1]}$, the portfolio choice $(q_a)_{a\in[0,1]}$, and the asset price $\tilde{p}$ such that they satisfy the following three conditions. First, given the information choice and the asset price, individual portfolio choice maximizes the expected utility (2.2.4) conditional on the realization of signals and price. Second, the asset price is set to clear the market. Lastly, taking the aggregates of agents' information choices as given, agent a's attention allocation $\sigma_{\theta,a}$ and $\sigma_{\zeta,a}$ maximize the expected utility (2.2.23), for almost every agent $a \in [0,1]$, conditional on prior information subject to information capacity constraint $(\sigma_{\theta,a}, \sigma_{\zeta,a}) \in K_a$ with rational expectation of an asset market equilibrium price $\tilde{p}$ and portfolio choice $(q_a)_{a\in[0,1]}$.

The first two conditions of the equilibrium amount to an asset market equilibrium; and since we only consider a pr-REE, such an asset market equilibrium can be characterized by some D that solves Equation (1.2.11). The last condition basically requires that, given that agents anticipate a pr-REE D in the asset market, the profile of attention allocation is a Nash equilibrium of the game of attention allocation. Therefore, the overall equilibrium is formally defined equivalently as follow.

Definition 1.3.1. An *overall equilibrium* of the game of attention allocation consists of a profile of attention allocation $(\sigma_{\theta,a}, \sigma_{\zeta,a})_{a\in[0,1]}$ and an asset market equilibrium D such that D solves equation (1.2.11) and condition (1.3.13) holds with the expected utility U_a^D corresponding to the asset equilibrium D.

Impossibility of Purely-Noisy REE under Attention Allocation

Asset price is possibly purely noisy in the asset market equilibrium when essentially no agents have private signals on the asset payoff but some agents have private signals on the noisy supply of the asset. However, when taking into account the choice of attention allocation, a purely noisy asset price cannot be supported in any overall equilibrium. Even when every agent choose to process the supply information, agents always coordinate on a pr-REE $D > 0$ so that the asset price is covarying with the asset payoff.

Note that the supply information is useful to agents only when the price is partially-revealing with $D > 0$. When the asset price is revealing some payoff information, agents can use the supply information combined with the asset price to improve their estimations of the payoff. However, when the asset price is purely noisy, agents cannot improve their estimation of the payoff by learning the supply. Hence, in this case, they will choose to learn the payoff directly if agents have the option to learn some payoff information. Formally, we say that an agent *a can learn the payoff* if there exist some $(\sigma_{\theta,a}, \sigma_{\zeta,a})$ in the information constraint set K_a such that $\sigma_{\theta,a} > 0$.[8]

Proposition 1.3.2. *In every overall equilibrium, the asset price is not purely noisy, i.e. $D \neq 0$, if and only if the set of agents who can learn the payoff has a strictly positive measure.*

This is reminiscent of the paradox of Grossman and Stiglitz (1980), which says that, if information of asset payoff is costly, price cannot perfectly reflect that information. However, here we establish that the asset price must reflect some information about the asset payoff if this information is freely available even though the agents have the option not to learn the payoff information. The intuition is simple. Suppose that the asset price is purely noisy, then it must be the case that essentially no one chooses to process the payoff information and has a private signal on the asset payoff. However, it is then more beneficial for every agent to directly learn the payoff information to improve her prediction of the expected return. But if so, then asset price cannot be purely noisy when some agents have private signals of the asset payoff. This reaches the contradiction.

[8] Conversely, an agent *a cannot learn the payoff* if $\sigma_{\theta,a} = 0$ for every $(\sigma_{\theta,a}, \sigma_{\zeta,a}) \in K_a$.

It turns out cumbersome to characterize the equilibria when agents are heterogeneous in such many dimensions and with a general information constraint. Hence, in the following sections, the equilibria are characterized under the information entropy constraint and agents are heterogenous in their prior beliefs.

1.4 Attention Allocation under Information Entropy

In this section, the information processing capacity is measured by information entropy. Hence, we call it the *game of attention allocation under information entropy.* Moreover, agents are assumed to have the same risk aversion and same information capacity, i.e. $\lambda_a = \lambda$, $H_a = H$. Varying individual parameters such as risk averse λ_a or information capacity H_a will only affect the asset market equilibrium D, which is taken as the same to all the agents in equilibrium. But individual decision depends crucially on the ratio of prior precisions between payoff and supply information. If the ratio of prior precisions, $\pi_{\theta,a}/\pi_{\zeta,a}$, vary across agents, then individual decision is most likely different from each other because the constraint sets vary across individuals.

To keep the exposition simple and clear, let us just consider one dimension of heterogeneity among agents: the prior precision $\pi_{\theta,a}$ of the payoff information varies across agents, and assume that $\pi_\theta \sim F(\pi_\theta)$ where F is a c.d.f. on an interval $[0, \infty)$. This heterogeneity is sufficient to generate difference of the ratio of prior precisions $\pi_{\theta,a}/\pi_\zeta$ among agents. The heterogeneity in prior precision ratio determines a "comparative advantage" in information for different agents: Some agents have higher precision ratio $\pi_{\theta,a}/\pi_\zeta$ if she has a high $\pi_{\theta,a}$, which means that *a priori* some agents have superior information about the fundamental value of the asset than others, while others are relatively more confident about their supply information.

We first obtain the asset market equilibrium with an arbitrary profile of attention allocation $(\sigma_{\theta,a}, \sigma_{\zeta,a})_{a\in[0,1]}$ by using the results in the Section 1.2. Denote the *average signal precisions* to be $\Sigma_\theta = \int_0^1 \sigma_{\theta,a} da$ and $\Sigma_\zeta = \int_0^1 \sigma_{\zeta,a} da$ and the *average prior precisions* to be $\Pi_\theta = \int_0^\infty \pi_\theta dF(\pi_\theta)$. Then the equilibrium of the asset market can be solved by replacing $\hat{\Sigma}_\theta = \lambda^{-1}\Sigma_\theta$, $\hat{\Sigma}_\zeta = \lambda^{-1}\Sigma_\zeta$, $\hat{\Pi}_\theta = \lambda^{-1}\Pi_\theta$, and $\hat{\Pi}_\zeta = \lambda^{-1}\pi_\zeta$ in equations (1.2.11)-(1.2.13). In particular, the equilibrium payoff informativeness D of the asset price solves

$$\Sigma_\zeta D^2 - \lambda D + \Sigma_\theta = 0. \tag{1.4.1}$$

1.4.1 Specialization of Learning

Each agent faces an information constraint of the form the information entropy. The action space of agent a is

$$K_a = \left\{ \sigma_{\theta,a}, \sigma_{\zeta,a} \geq 0 \,\middle|\, \frac{\pi_{\theta,a} + \sigma_{\theta,a}}{\pi_\theta} \frac{\pi_\zeta + \sigma_{\zeta,a}}{\pi_\zeta} \leq H \right\}. \tag{1.4.2}$$

There are several reasons for us to choose information entropy as our primary form of the information constraint. Besides the aforementioned axiomatic foundation, the entropy-type constraint is widely used in the economics, econometrics, and statistics. Moreover, the information capacity is measured by the ratio of precisions of prior and posterior beliefs, which extends the measurement of the quality of information used by Grossman and Stiglitz (1980). Van Nieuwerburgh and Veldkamp (2010) also argue that this type of information constraint is scale neutral and empirical evidence matches the prediction of agents' information choice in the financial market under the entropy-type constraint.

Information has increasing returns to its scale under information entropy. The higher the prior precision of one type of information, the larger the maximum posterior precision of that type of information can be learned. This have an intuitive interpretation: given the information capacity or the upper limit of information processing capability, if the agent is initially more knowledgeable about one type of information, say the payoff information, then it just requires little of her effort to digest new piece of information on the fundamental value of the asset and so it is more effective for her to reduce the uncertainty in the payoff information. Graphically, the complement of the action space K_a is a convex subset in $\mathbb{R}^2_+$ as showed in the left panel of Figure 1.3.1.

By Proposition 2.3.1, the individual problem is to maximize the posterior precision subject to the information constraint, which is linear in the signal precisions chosen by the agent. Therefore, the optimum must be attained at the end point of the constraint set.[9] In equilibrium, individual agent *specializes of learning*, i.e. chooses to process just one type of the information. Agent a chooses to *learn the payoff* or be a *fundamentalist* if she chooses $(\sigma_{\theta,a}, \sigma_{\zeta,a}) = ((H-1)\pi_{\theta,a}, 0)$, and chooses to *learn the supply* or be a *chartist* if $(\sigma_{\theta,a}, \sigma_{\zeta,a}) = (0, (H-1)\pi_\zeta)$. Agents' choices depend on how informative the price is revealing the payoff information in the asset market.

Proposition 1.4.1. *Let an asset market equilibrium D be given. Agent a chooses to learn the payoff, $\sigma_{\zeta,a} = 0$, if $\pi_{\theta,a}/\pi_\zeta > D^2$ and chooses to learn the supply, $\sigma_{\theta,a} = 0$, if $\pi_{\theta,a}/\pi_\zeta < D^2$. If $\pi_{\theta,a}/\pi_\zeta = D^2$, then agent a is indifferent between learning the payoff or supply.*

The proof of the proposition is omitted since it is a direct corollary to the Proposition 2.3.1. For two end points of the constraint set, the rate of exchange of one unit precision of payoff signal to that of the supply signal is $\pi_{\theta,a}/\pi_\zeta$. Each unit of

[9] One of the most important differences among various specifications of information constraints is whether the optimal solution to the information choice problem is at the corners of the constraint set. This determines whether "specialized learning", termed by Van Nieuwerburgh and Veldkamp (2010), emerges in the equilibrium. Van Nieuwerburgh and Veldkamp (2010) show that there can be "specialized learning" when the information constraint of entropy-type or additive; Farboodi and Veldkamp (2016) show that agents will choose to learn fundamental information when the information capacity is low with their quadratic information constraint. We will simply focus on the entropy-type constraint and argue for the specialized learning in the equilibrium of attention allocation.

incremental of the precisions of the payoff and supply signal lead to one and D^2 units, respectively, of incremental in the posterior precision, that is, the (marginal) rate of substitution of one unit precision of payoff signal to that of the supply signal is $1/D^2$. Hence, the comparison between rate of exchange and rate of substitution determines the individual choice.

1.4.2 All-Fundamentalist and All-Chartist Equilibrium

Agents specialize their learning under information entropy. If all the agents choose to learn the payoff in equilibrium, then it is called an *all-fundamentalist equilibrium.* Correspondingly, an *all-chartist equilibrium* is such that all the agents choose to learn the supply.

The existence of all-fundamental and all-chartist equilibria requires that the support of F is bounded, i.e. there are lower bound and upper bound for an agent's prior precision of payoff information. The infimum and supremum of supp$F$ are denoted by $\underline{\pi}_\theta$ and $\bar{\pi}_\theta$, respectively, where $\bar{\pi}_\theta$ can be infinity. For there existing an all-fundamentalist equilibrium, every agent chooses to learn the payoff, and hence, so does the agents with lowest prior precision of the payoff information. This requires that the infimum $\underline{\pi}_\theta$ of supp$F$ to be high enough. However, in an all-chartist equilibrium, the possibly highest prior precision $\bar{\pi}_\theta$ should be low enough so that every agent, including the agents with highest prior precision, choose to learn the supply. The following proposition characterizes the existence of all-fundamentalist and all-chartist equilibria.

Proposition 1.4.2. *Suppose that supp$F$ is bounded, and let $\underline{\pi}_\theta$ and $\bar{\pi}_\theta$ be its infimum and supremum, respectively.*

(i) There exists an all-fundamentalist equilibrium if and only if

$$\Pi_\theta^2 \pi_\zeta / \underline{\pi}_\theta \leq \lambda^2/(H-1)^2. \tag{1.4.3}$$

The corresponding asset market equilibrium must be a GS equilibrium.

(ii) There exists an all-chartist equilibrium if and only if

$$\bar{\pi}_\theta \pi_\zeta \leq \lambda^2/(H-1)^2. \tag{1.4.4}$$

The corresponding asset market equilibrium must be a COM equilibrium.

Inequality (1.4.3) is equivalent to say that $\underline{\pi}_\theta$ is at least $\Pi_\theta \pi_\zeta/\lambda^2/(H-1)^2$ and inequality (1.4.4) is the same as $\bar{\pi}_\theta \leq \lambda^2/\pi_\zeta(H-1)^2$. If $\underline{\pi}_\theta = \bar{\pi}_\theta = \pi_\theta$, then $\Pi_\theta = \pi_\theta$ and both inequalities (1.4.3) and (1.4.4) reduce to the same inequality $\pi_\theta \pi_\zeta \leq \lambda^2/(H-1)^2$.

Note that the corresponding asset market equilibrium is a GS one in the all-fundamentalist equilibrium and COM in the all-chartist equilibrium. First, this result confirms Theorem 1.3.2. Even though it is possible that there is no one has private signals on the payoff, the payoff informativeness is strictly positive and the asset price is still covarying with the asset payoff in an all-chartist equilibrium. Moreover, the properties of the asset price is drastically different in an all-fundamentalist equilibrium and an all-chartist one.

Corollary 1.4.1. *The payoff informativeness D^{GS} of an all-fundamentalist equilibrium is increasing in the information capacity H but decreasing in the risk aversion λ. The payoff informativeness D^{COM} of an all-chartist equilibrium is decreasing in the information capacity H but increasing in the risk aversion λ.*

The comparative statics of the two types of equilibria is almost opposite to each other. This is due to the following fact: In each type of equilibrium, the corresponding asset market equilibrium is unique; in the all-fundamentalist equilibrium, the asset market equilibrium is a GS equilibrium, i.e. payoff informativeness is increasing in Σ_θ, while in the all-chartist equilibrium, the asset market equilibrium is a COM equilibrium.

From the empirical point of view, although it is difficult to rule out the situation where individual traders herd to one type of information in some extreme cases, it is also quite often to see that agents try to differentiate their sources of information and choose to learn different types of information. Moreover, there cannot be any all-fundamentalist and all-chartist equilibria especially when the supp$F$ is unbounded or it includes 0. Inequalities (1.4.3) cannot hold when $\underline{\pi}_\theta = 0$ and inequalities (1.4.4) cannot hold when $\bar{\pi}_\theta$ tends to infinity.

1.4.3 Separating Equilibrium

Nevertheless there is another possibility of the equilibrium: agents differentiates their specialization of learning and fundamentalists and chartists coexist, which we call it *separating equilibrium* because agents can be separated into two groups, fundamentalists and chartists. In Section 1.4.3 and Section 1.4.3, the separating equilibrium in two different settings are studied respectively: Firstly, there are only two types of agents, one with high prior precision of payoff information (*high type*) and the other with a low precision (*low type*). The general pattern of separating equilibrium is such that high types choose to learn the payoff and low types choose to learn the supply. Secondly, in the model with continuum of agents, we further study how the cutoff of specialization endogenously determined in the equilibrium.

Two Types of Agents

First, suppose that there are only two values of the prior precision $\pi_{\theta,a}$ among agents. Given some $\alpha \in (0,1)$, if $a \leq \alpha$, then $\pi_{\theta,a} = \bar{\pi}_\theta$, otherwise, $\pi_{\theta,a} = \underline{\pi}_\theta$, with $\bar{\pi}_\theta > \underline{\pi}_\theta$. Therefore, the measure of *high types*, i.e. agents with $\pi_{\theta,a} = \bar{\pi}_\theta$, is α, and measure of *low types*, i.e. agents with $\pi_{\theta,a} = \underline{\pi}_\theta$, is $1-\alpha$. Then, $\Pi_\theta = \alpha\bar{\pi}_\theta + (1-\alpha)\underline{\pi}_\theta$.

In this case, it is sufficient to specify the strategy for each type to characterize the equilibrium. First, Proposition 2.3.2 can be applied here and the infimum and supremum of supp$F$ are $\underline{\pi}_\theta$ and $\bar{\pi}_\theta$, respectively. There is an all-fundamentalist equilibrium if and only if inequality (1.4.3) holds, and there is an all-chartist equilibrium if and only if inequalities (1.4.4) holds.

Besides the all-fundamentalist and all-chartist equilibria, there can be another *separating equilibrium* where two types take different actions. In particular, it must be the case that high types choose to learn the payoff and low types specialize in the learning the supply. Suppose that, in a separating equilibrium, conversely high types choose to learn the supply, which implies that $\bar{\pi}_\theta/\pi_\zeta \leq D^2$, and low types choose to learn the payoff, which implies that $\underline{\pi}_\theta/\pi_\zeta \geq D^2$. It further implies that $\underline{\pi}_\theta \geq \bar{\pi}_\theta$ by transitivity, contradicting with the our assumption $\bar{\pi}_\theta > \underline{\pi}_\theta$.

Consider a separating strategy profile such that high types choose to learn the payoff and low types specialize in the learning the supply. Then $\Sigma_\theta = \alpha(H-1)\bar{\pi}_\theta$ and $\Sigma_\zeta = (1-\alpha)(H-1)\pi_\zeta$, and D^{GS} and D^{COM} can be computed accordingly. Since $D^{GS} < D^{COM}$, such a strategy profile can be supported in an equilibrium if and only if one of the followings holds: (i) $D^{GS} \leq \sqrt{\underline{\pi}_\theta/\pi_\zeta} \leq D^{COM} \leq \sqrt{\bar{\pi}_\theta/\pi_\zeta}$, (ii) $\sqrt{\underline{\pi}_\theta/\pi_\zeta} \leq D^{GS} \leq \sqrt{\bar{\pi}_\theta/\pi_\zeta} \leq D^{COM}$, or (iii) $\sqrt{\underline{\pi}_\theta/\pi_\zeta} \leq D^{GS} < D^{COM} \leq \sqrt{\bar{\pi}_\theta/\pi_\zeta}$. We then have the following proposition after some algebra.

Proposition 1.4.3. *There exists a separating equilibrium where high types choose to learn the payoff and low types choose to learn the supply if and only if one of the following holds:*

(i) $\frac{\Pi_\theta^2 \pi_\zeta}{\underline{\pi}_\theta} \leq \frac{\lambda^2}{(H-1)^2} \leq \bar{\pi}_\theta \pi_\zeta$*; or*

(ii) $\bar{\pi}_\theta \pi_\zeta \leq \frac{\lambda^2}{(H-1)^2} \leq \frac{\Pi_\theta^2 \pi_\zeta}{\underline{\pi}_\theta}$*; or*

(iii) $4\alpha(1-\alpha)\bar{\pi}_\theta \pi_\zeta \leq \frac{\lambda^2}{(H-1)^2} < \min\left\{\frac{\Pi_\theta^2 \pi_\zeta}{\underline{\pi}_\theta}, \bar{\pi}_\theta \pi_\zeta\right\}$ *and* $\frac{\underline{\pi}_\theta}{\underline{\pi}_\theta + \bar{\pi}_\theta} < \alpha < \frac{1}{2}$.

In cases (i) and (ii), there is only one separating equilibrium. There must be a COM equilibrium in the separating equilibrium in case (i), and a GS equilibrium in case (ii). However, for the very special case (iii), there are two separating equilibria with the same attention allocation but different asset market equilibria.

Let us consider some numerical examples of the two-type model.

Example 1.4.1. Let $\underline{\pi}_\theta = 0.2$, $\bar{\pi}_\theta = 1.2$, $\pi_\zeta = 1$, and $H = 2$. Moreover, consider the following:

(i) $\alpha = 0.2$ and $\lambda = 1.05$. Then we have $\frac{\Pi_\theta^2 \pi_\zeta}{\underline{\pi}_\theta} \leq \frac{\lambda^2}{(H-1)^2} \leq \bar{\pi}_\theta \pi_\zeta$. Thus, there are two equilibria: an all-fundamentalist equilibrium and a separating equilibrium.

(ii) $\alpha = 0.3$ and $\lambda = 1.1$. Then we have $\bar{\pi}_\theta \pi_\zeta \leq \frac{\lambda^2}{(H-1)^2} \leq \frac{\Pi_\theta^2 \pi_\zeta}{\underline{\pi}_\theta}$. Again there are two equilibria in this configuration of parameters, but one is an all-chartist equilibrium and the other is a separating equilibrium.

(iii) $\alpha = 0.3$ and $\lambda = 1.05$. Then we have $4\alpha(1-\alpha)\bar{\pi}_\theta \pi_\zeta \leq \frac{\lambda^2}{(H-1)^2} < \min\left\{\frac{\Pi_\theta^2 \pi_\zeta}{\underline{\pi}_\theta}, \bar{\pi}_\theta \pi_\zeta\right\}$ and $\frac{\underline{\pi}_\theta}{\underline{\pi}_\theta + \bar{\pi}_\theta} < \alpha < \frac{1}{2}$. In this case, the equilibrium must be separating, i.e. the high type specialize to learn the payoff signal and the low type learn the supply signal in the stage of attention allocation, but there could be two corresponding asset market equilibria. Hence there are two separating (overall) equilibria.

(iv) $\alpha = 0.7$ and $\lambda = 1.05$. Then we have $4\alpha(1-\alpha)\bar{\pi}_\theta \pi_\zeta \leq \frac{\lambda^2}{(H-1)^2} < \min\left\{\frac{\Pi_\theta^2 \pi_\zeta}{\underline{\pi}_\theta}, \bar{\pi}_\theta \pi_\zeta\right\}$ but $\alpha > \frac{1}{2}$. No equilibrium.

(v) $\alpha = 0.3$ and $\lambda = 1.2$. Then we have $\max\left\{\frac{\Pi_\theta^2 \pi_\zeta}{\underline{\pi}_\theta}, \bar{\pi}_\theta \pi_\zeta\right\} \leq \frac{\lambda^2}{(H-1)^2}$. In this case, there are two pooling equilibria, an all-fundamentalist and an all-chartist equilibrium.

Continuum of Types of Agents

In Section 1.4.3, if there is some heterogeneity in agents' prior precisions, then there can be an equilibrium where different types of agents choose to learn different types of information. It is more interesting to further ask how high the prior precision should be so that agents will choose to learn the payoff if the distribution of prior precisions are continuous.

Therefore, consider a continuum of types of agents, that is, the support of F is a continuum. Specifically, let $\text{supp}F = [\underline{\pi}_\theta, \bar{\pi}_\theta) \subseteq [0, \infty)$ where $\text{supp}F$ can be bounded, $\bar{\pi}_\theta < \infty$, or unbounded, $\bar{\pi}_\theta = \infty$. An agent with prior precision $\pi_{\theta,a}$ of the payoff information is said to be a *type* $\pi_{\theta,a}$. Assume F has a continuous density $f = F'$. First, there are all-fundamentalist and all-chartist equilibria are as described in Proposition 2.3.2 if $\text{supp}F$ is bounded.

The separating equilibrium in this case is more interesting. As we have seen in Section 1.4.3, high types choose to learn the payoff and low types choose to learn the supply. Because of the linearity of the objective function and shape of the information entropy constraint set, the separating equilibrium is monotone. If some type choose to learn the payoff, then all higher types will choose to learn the payoff, and conversely, if some type choose to learn the supply, then all lower types will choose to learn the supply. Then the following lemma is established consequently.

Lemma 1.4.1. *In a separating equilibrium, there is a cutoff type π_θ^* such that all agents with higher types $\pi_{\theta,a} > \pi_\theta^*$ choose to learn the payoff and all agents with lower types $\pi_{\theta,a} < \pi_\theta^*$ choose to learn the supply.*

Remark 1.4.1. This cutoff type π_θ^* is endogenously determined, who is indifferent to learn either the payoff or the supply, in the equilibrium. Therefore, we just call such a type π_θ^* a *separating equilibrium cutoff.*[10]

[10]Given a cutoff type $\pi_\theta^* \in (\underline{\pi}, \bar{\pi})$, we can construct two equilibria: in one equilibrium, types $\pi_{\theta,a} \geq \pi_\theta^*$ choose to learn the payoff and types $\pi_{\theta,a} < \pi_\theta^*$ choose to learn the supply; in the other equilibrium, types $\pi_{\theta,a} > \pi_\theta^*$ choose to learn the payoff and types $\pi_{\theta,a} \leq \pi_\theta^*$ choose to learn the supply. These two equilibria differ only on a measure zero set, so we do not distinguish these two equilibria.

Suppose that π_θ^* is a separating equilibrium cutoff. The average precisions of signals can be calculated as

$$\Sigma_\theta = (H-1)\int_{\pi_\theta^*}^{\bar{\pi}_\theta} \pi_\theta dF(\pi_\theta) \tag{1.4.5}$$

$$\Sigma_\zeta = (H-1)\pi_\zeta F(\pi_\theta^*). \tag{1.4.6}$$

In equilibrium with cutoff π_θ^*, the agent with prior precision π_θ^* is indifferent of choosing what types of information to learn, which implies that, it is either

$$D^{COM} = \frac{\lambda + \sqrt{\lambda^2 - 4\Sigma_\theta\Sigma_\zeta}}{2\Sigma_\zeta} = \sqrt{\frac{\pi_\theta^*}{\pi_\zeta}} \text{ or} \tag{1.4.7}$$

$$D^{GS} = \frac{\lambda - \sqrt{\lambda^2 - 4\Sigma_\theta\Sigma_\zeta}}{2\Sigma_\zeta} = \sqrt{\frac{\pi_\theta^*}{\pi_\zeta}}. \tag{1.4.8}$$

Rearranging both equations yields that

$$\frac{\lambda}{H-1}\sqrt{\frac{\pi_\theta^*}{\pi_\zeta}} = \int_{\pi_\theta^*}^{\bar{\pi}_\theta} \pi_\theta dF(\pi_\theta) + \pi_\theta^* F(\pi_\theta^*). \tag{1.4.9}$$

We define the following function G by

$$G(\pi) = \int_{\pi}^{\bar{\pi}_\theta} \pi_\theta dF(\pi_\theta) + \pi F(\pi) - \frac{\lambda}{H-1}\sqrt{\frac{\pi}{\pi_\zeta}}. \tag{1.4.10}$$

The separating equilibrium cutoff π_θ^* has to be a root of $G(\pi) = 0$. Conversely, the proof of Lemma 1.4.2 in the Appendix to Chapter 1 shows that π_θ^* is a real root of $G(\pi) = 0$ for $\pi \in (\underline{\pi}_\theta, \bar{\pi}_\theta)$ is also a sufficient condition for π_θ^* to be a separating equilibrium.

Lemma 1.4.2. $\pi_\theta^* \in [\underline{\pi}_\theta, \bar{\pi}_\theta]$ *is a separating equilibrium cutoff if and only if it is a real root of* $G(\pi) = 0$.

Note that G is a convex function. To see this, the second order derivative of G,

$$G''(\pi) = f(\pi) + \frac{\lambda}{4(H-1)\sqrt{\pi_\zeta}}\pi^{-\frac{3}{2}} > 0. \tag{1.4.11}$$

This property gives rise to some interesting corollaries. First, two sufficient conditions for the existence of a unique separating equilibrium is exactly the same as (i) and (ii) in Proposition 2.3.3.

Proposition 1.4.4. *If (i)* $\bar{\pi}_\theta \pi_\zeta < \lambda^2/(H-1)^2 < \Pi_\theta \pi_\zeta / \underline{\pi}_\theta$ *or (ii)* $\Pi_\theta \pi_\zeta / \underline{\pi}_\theta < \lambda^2/(H-1)^2 < \bar{\pi}_\theta \pi_\zeta$, *then there exists a unique separating equilibrium.*

Second, in the extreme case when supp$F = \mathbb{R}_+$, there cannot be any all-fundamentalist and all-chartist equilibrium, and thus the equilibrium must be separating if there is any equilibrium. Moreover, $G(0) = \Pi_\theta > 0$ and $\lim_{\pi\to\infty} G(\pi) = \infty$. The convexity of G guranteens that it admits a unique minimizer, denoted by $\mathring{\pi}_\theta$, which solves the first-order condition

$$F(\mathring{\pi}_\theta) - \frac{\lambda}{2(H-1)\sqrt{\pi_\zeta \mathring{\pi}_\theta}} = 0. \tag{1.4.12}$$

Proposition 1.4.5. *Suppose that supp$F = \mathbb{R}_+$. Let $\mathring{\pi}_\theta$ be the unique minimizer of G. Then,*

(i) *If $G(\mathring{\pi}_\theta) > 0$, then there is no equilibrium.*

(ii) *If $G(\mathring{\pi}_\theta) = 0$, then there is a unique equilibrium and it is a separating equilibrium with cutoff $\mathring{\pi}_\theta$.*

(iii) *If $G(\mathring{\pi}_\theta) < 0$, then there are two equilibria and both are separating equilibria.*

Remark 1.4.2. Suppose there are two separating equilibria, denote the larger one by $\pi_\theta^{*,COM}$ and the smaller one by $\pi_\theta^{*,GS}$, i.e. $\pi_\theta^{*,COM} > \pi_\theta^{*,GS}$. The separating equilibrium $\pi_\theta^{*,COM}$ correspond to a COM asset market equilibrium $D^{COM} = \sqrt{\pi_\theta^{*,COM}/\pi_\zeta}$, while the separating equilibrium $\pi_\theta^{*,GS}$ correspond to a GS asset market equilibrium $D^{GS} = \sqrt{\pi_\theta^{*,GS}/\pi_\zeta}$.

In fact, Proposition 1.4.5 (ii) is a very special case, and except that, in (iii) we can observe the existence of multiple equilibria π_θ^*'s and the pattern of specialization in learning: high types with $\pi_{\theta,a} \geq \pi_\theta^*$ choose to learn the payoff signals and low types $\pi_{\theta,a} < \pi_\theta^*$ focus on learning the supply signal.

Finally, we examine some numerical examples of continuum-type models.

Example 1.4.2. Let $\pi_\zeta = 1$ and $H = 2$. Consider that the distribution of prior precision is uniform on $[0.2, 1.2)$. For uniform distribution it is sure that $\bar{\pi}_\theta \pi_\zeta < \frac{\Pi_\theta \pi_\zeta}{\underline{\pi}_\theta}$, and, specifically, $\frac{\Pi_\theta \pi_\zeta}{\underline{\pi}_\theta} = 2.45$ and $\bar{\pi}_\theta \pi_\zeta = 1.2$. So if $\sqrt{1.2} \leq \lambda < \sqrt{2.45}$, then there is an all-chartist equilibrium and a separating equilibrium. Moreover, when $\lambda \geq \sqrt{2.45}$, then two pure equilibria co-exist. If $\lambda < \sqrt{1.2}$, then there will be no pure equilibrium, and the separating equilibrium π_θ^* satisfies the following equation

$$0.5\pi_\theta^* - 0.2\pi_\theta^* + 0.72 - \lambda\sqrt{\pi_\theta^*} = 0. \tag{1.4.13}$$

Example 1.4.3. Let $\pi_\zeta = 1$ and $H = 2$. Consider that the distribution of prior precision is exponential with mean Π_θ, i.e. $F(\pi_\theta) = 1 - e^{-\pi_\theta/\Pi_\theta}$. If there exists any equilibrium, then it must be separating equilibrium, and the equilibrium π_θ^* satisfies the following equation

$$\pi_\theta^* + \Pi_\theta e^{-\pi_\theta^*/\Pi_\theta} - \lambda\sqrt{\pi_\theta^*} = 0. \tag{1.4.14}$$

Comparative Statics of Separating Equilibria. We study the comparative statics of the separating equilibrium, in particular, when $\mathrm{supp}F = \mathbb{R}_+$ and $G(\mathring{\pi}_\theta) < 0$, the case (iii) in Proposition 1.4.5. Under these assumptions, there can be only two separating equilibria. Note that the cutoff π_θ^* of a separating equilibrium characterizes not only the attention allocation strategy in equilibrium but also the corresponding asset market equilibrium $D = \sqrt{\pi_\theta^*/\pi_\zeta}$.

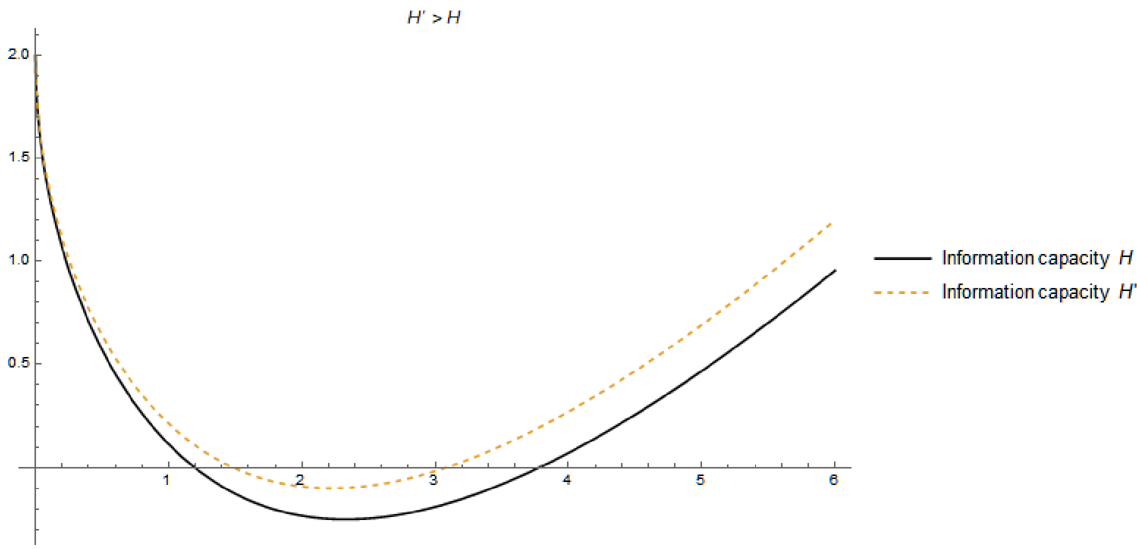

Figure 1.4.1: Effects of changing information capacity H

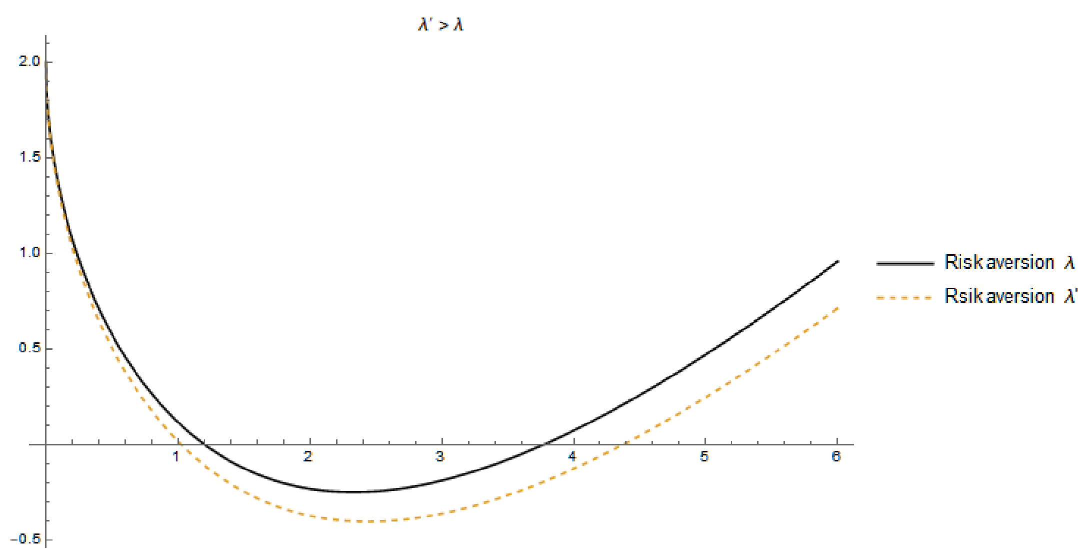

Figure 1.4.2: Effects of changing risk aversion λ.

Proposition 1.4.6. *Let $\pi_\theta^{*,COM}$ and $\pi_\theta^{*,GS}$ be two separating equilibria defined as in remark 1.4.2 in the same economy. Then $\pi_\theta^{*,COM}$ is decreasing in information capacity H but increasing in risk aversion λ, and $\pi_\theta^{*,GS}$ is increasing in information capacity H but decreasing in risk aversion λ.*

To see Proposition 1.4.6 more clearly, look at the Figure 1.4.1 and 1.4.2. Increasing information capacity from H to H' moves the curve G upwards, therefore, the larger root of G shifts left and smaller root right given the shape of curve G, while increasing risk aversion from λ to λ' has the exactly opposite effects.

Remark 1.4.3. Each separating equilibrium π_θ^* also characterizes its corresponding asset market $D = \sqrt{\pi_\theta^*/\pi_\zeta}$. Therefore, for the larger separating equilibrium π_θ^*, its corresponding COM asset market equilibrium D^{COM} is decreasing in information capacity H but increasing in risk aversion λ, while, for the smaller separating equilibrium π_θ^*, it is exactly opposite for the corresponding GS asset market equilibrium D^{GS}. This observation mirrors the same properties, Corollary 1.4.1, of all-fundamentalist and all-chartist equilibria.

1.5 Existence of Equilibrium

In this section, we return to the model in Section 1.3 and discuss the existence of equilibrium in general. For a fixed pr-REE D, the mapping $a \mapsto U_a^D$ is measurable since we assume the mapping from a to the individual characteristics to be measurable; and hence it defines a game with a continuum of atomless agents (e.g. Schmeidler (1973), Mas-Colell (1984), and Rath (1992)) or an *aggregative game*, where each agent's expected utility only depends on her own strategy and some aggregates of the strategies of all the agents, $\hat{\Sigma}_\theta$ and $\hat{\Sigma}_\zeta$ in this case.

Here we follow the approach of Rath (1992). Define a strategy profile $(\sigma_{\theta,\cdot}, \sigma_{\zeta,\cdot})$ to be a measurable function from $[0, 1]$ to the compact set K_a. We define two other sets:

$$
\begin{aligned}
S &= \int_0^1 \lambda_a^{-1} K_a da \\
&= \left\{ \left(\hat{\Sigma}_\theta, \hat{\Sigma}_\zeta\right) | (\sigma_{\theta,a}, \sigma_{\zeta,a}) \in K_a,\ \forall a \right\} && (1.5.1) \\
T &= \left\{ \left(\hat{\Sigma}_\theta, \hat{\Sigma}_\zeta\right) \in S \middle| \hat{\Sigma}_\theta \hat{\Sigma}_\zeta \leq 1/4 \right\}. && (1.5.2)
\end{aligned}
$$

The set S is the set of all possible aggregates $\hat{\Sigma}_\theta$ and $\hat{\Sigma}_\zeta$ given individual agent's action space K_a, while set T is the set of aggregates where there exists some pr-REE so that the expected utility function $U_a^D\left(\sigma_{\theta,a}, \sigma_{\zeta,a}; \cdot, \cdot\right)$ are defined for each agent a.

The best response correspondence $B^\ell : [0,1] \times T \rightrightarrows K_a$, $\ell \in \{GS, COM\}$ is

$$
\begin{aligned}
B^\ell\left(a; \hat{\Sigma}_\theta, \hat{\Sigma}_\zeta\right) = \Big\{ (\sigma_{\theta,a}, \sigma_{\zeta,a}) \in K_a \Big| \; & D = D^\ell \text{ and} && (1.5.3) \\
& U_a^D\left(\sigma_{\theta,a}, \sigma_{\zeta,a}; \hat{\Sigma}_\theta, \hat{\Sigma}_\zeta\right) \geq U_a^D\left(\sigma_\theta, \sigma_\zeta; \hat{\Sigma}_\theta, \hat{\Sigma}_\zeta\right), \forall \left(\sigma_\theta, \sigma_\zeta\right) \in K_a \Big\}.
\end{aligned}
$$

Moreover, define a correspondence $\Gamma^\ell : T \to S$, $\ell \in \{GS, COM\}$ by

$$
\Gamma^\ell\left(\hat{\Sigma}_\theta, \hat{\Sigma}_\zeta\right) = \int_0^1 \lambda_a^{-1} B^\ell\left(a; \hat{\Sigma}_\theta, \hat{\Sigma}_\zeta\right) da, \tag{1.5.4}
$$

which is the set of actual aggregates $\int_0^1 \lambda_a^{-1}\sigma_{\theta,a} da$ and $\int_0^1 \lambda_a^{-1}\sigma_{\zeta,a} da$ of the best responses to the conjectured aggregates $\left(\hat{\Sigma}_\theta, \hat{\Sigma}_\zeta\right)$. Note that, if Γ has a fixed point $\left(\hat{\Sigma}_\theta^*, \hat{\Sigma}_\zeta^*\right)$, i.e. the actual aggregates coincide with the conjecture, then there exist a strategy profile $\left(\sigma_{\theta,\cdot}^*, \sigma_{\zeta,\cdot}^*\right)$ such that $\hat{\Sigma}_\theta^* = \int_0^1 \lambda_a^{-1}\sigma_{\theta,a}^* da$ and $\hat{\Sigma}_\zeta^* = \int_0^1 \lambda_a^{-1}\sigma_{\zeta,a}^* da$ and $\left(\sigma_{\theta,a}^*, \sigma_{\zeta,a}^*\right) \in B^\ell\left(a; \hat{\Sigma}_\theta^*, \hat{\Sigma}_\zeta^*\right)$ for almost every $a \in [0,1]$, and $D = D^{GS}$ or D^{COM} given $\left(\hat{\Sigma}_\theta^*, \hat{\Sigma}_\zeta^*\right)$ so it solves the Equation (1.2.11). Hence $\left(\sigma_{\theta,\cdot}^*, \sigma_{\zeta,\cdot}^*\right)$ and D constitute an overall equilibrium.

Following the proof of Theorem 2 in Rath (1992), we can show that B^ℓ is nonempty valued and $B^\ell(a; \cdot, \cdot)$ has a closed graph; and $B^\ell\left(\cdot; \hat{\Sigma}_\theta, \hat{\Sigma}_\zeta\right)$ admits a measurable selection. Consequently, Γ^ℓ is non-empty- and convex-valued and has a closed graph. However, to evoke Kakutani's fixed point theorem for the correspondence Γ, it has to hold that $T = S$, otherwise the existence of equilibrium cannot be guaranteed. These results are summarized in the following lemma.

Lemma 1.5.1. *There exist an equilibrium of the game of attention allocation if and only if the correspondence Γ^ℓ, $\ell \in \{GS, COM\}$, defined by (1.5.4) has a fixed point.*

The existence of the equilibrium can be guaranteed if each agent a (except for a null set) are sufficiently risk averse and $\{K_a\}_{a\in[0,1]}$ has an essential supremum so that $\hat{\Sigma}_\theta \hat{\Sigma}_\zeta \leq 1/4$ holds.

Theorem 1.5.1. *Suppose that λ_a is bounded below by $\underline{\lambda} > 0$ and K_a is bounded above by $(\bar{\sigma}_\theta, \bar{\sigma}_\zeta)$ for almost every $a \in [0,1]$. There exists an equilibrium of the game of attention allocation if $\underline{\lambda} \geq 2\sqrt{\bar{\sigma}_\theta \bar{\sigma}_\zeta}$.*

The other interpretation of the sufficient condition $\underline{\lambda} \geq 2\sqrt{\bar{\sigma}_\theta \bar{\sigma}_\zeta}$ is that the information limits are small enough so that it ensure that $S = T$ to apply the fixed point theorem. However, $S = T$ is only a sufficient condition because it is possible for Γ^ℓ admits a fixed point even though $T \subset S$. Moreover, because there can be two different asset market equilibrium D^{COM} and D^{GS}, there can be a fixed point for Γ^{GS} and another fixed point for Γ^{COM}, so typically there will be at least two overall equilibria: one equilibrium consists of a COM equilibrium, and the other of a GS equilibrium.

1.6 Conclusion and Implications

In this study, we consider an environment where asset price is affected by both its fundamental value and market conditions. The payoff and supply information are both learnable to the agents. Typically the asset market will admit multiple equilibria.

In the rational expectation equilibrium, the price aggregates individual choices in this environment and also serves a public signal, and hence the aggregation of information imposes some externalities on individuals. Knowing that the aggregation of information choices would affect the payoff informativeness of the asset price, the agents have to take it into account when they are acquiring information. The multiplicity of equilibria is transmitted from the asset market to the game of attention allocation. It is common to see that multiple equilibria can exist in the attention allocation game.

We establish some key insights from the model under information entropy constraint. Agents' difference in prior precisions, indicating different endowed information sources or estimation procedures, leads to a general pattern of specialized learning emerges: typically agents with relatively higher prior precision about the payoff information choose to the learn the payoff signals, and the rest learn the supply signals. There can be multiple equilibria. In particular, when there is a continuum of unbounded types of agents, the payoff informativeness of the asset price is higher in an equilibrium where the number of fundamentalists is fewer.

Farboodi and Veldkamp (2016) document a secular shift in financial analysis from fundamental analysis (learning the payoff) to demand analysis (learning the supply) and, at the same time, the payoff informativeness is increasing over time. They argue that this phenomenon is a consequence of increasing information capacity. However, their results crucially depend on their assumption of concave information constraints. Because of diminishing return of the payoff information, the incentive of processing additional payoff information is weakened over time and each individual chooses to learn more about the supply.

In this paper, we can offer an alternative explanation of this secular shift in financial analysis as a consequence of increasing information capacity. Suppose that agents coordinate on the GS asset market equilibrium. Then, in a separating equilibrium, the cutoff, i.e. the lowest prior precision of the payoff information for agents to choose to learn the payoff, is increasing so that the some of the fundamentalists switches to the chartists. Therefore, there are more traders, e.g. retailing investors, hedge fund managers, etc., choose to process the supply information (non-fundamental data). Moreover, the payoff informativeness of the asset price increases as argued in Remark 1.4.3.

However, there are also some different empirical implications between this paper and Farboodi and Veldkamp (2016). According to their argument, as information technology improves and information capacity increases, each trader chooses to use more demand analysis (learn more about the supply). However, in our model, agents do not change their specialization of learning, which crucially depends on their prior precisions, if their individual information capacity increases. The increase in information capacity has an equilibrium effect leading to that some agents switch their specialization of learning because the payoff informativeness of the asset price also increases. Therefore, a close examination whether individual investors, e.g. fund managers, retailing traders, specialize in certain investment strategy can distinguish different theories.

Chapter 2

Strategic Timing in Information Acquisition

2.1 Introduction

Information is crucial in the financial markets. However, traders are not born with information. Acquiring information increases traders' expected profits, but it is costly. Moreover, the cost and benefit of acquiring information can change over time, as does the incentive for acquiring information. Traders evaluate the benefit against the cost of acquiring information according to their prior knowledge and the evolution of market conditions.

The literature on information acquisition in financial markets starts with Grossman and Stiglitz (1980). While many studies relate to static information acquisition, Banerjee and Breon-Drish (2016) extend to how an individual acquires information dynamically as a problem of exercising a real option in a Kyle's (1985) trading mechanism, and they ask when does a trader decide to become informed. Similar to irreversible investments, traders delay the acquisition of information. The trader will only take action when the expected payoff is higher than what would be necessary to cover the costs.

When multiple traders can become informed, the timing of their information acquisition is further complicated by competition. On the one hand, information reduces uncertainty and increases the informed traders' expected profit. This is especially true when one trader acquires information earlier than the other. The trader can enjoy an informational advantage and experience reduced competition in trading with other traders so that they can trade aggressively and earn more than anticipated profits. On the other hand, while one trader pays the cost of acquiring information, some information can be disclosed to another trader who does not need to pay a penny. Uninformed traders can wait for free access to the information for which other traders have paid.

In this paper, we study when traders choose to be informed if they are competing for acquiring information. In the model, the trading decisions are simple so that attention can be concentrated on the timing decisions between traders. Time is continuous and asset prices evolves over time. Without acquiring private information, traders are indifferent as to whether or not they trade. Only after uncovering some piece of information, and when the private belief is divergent from the public belief, will the trader choose to trade in the favored direction up to their *trading capacity*, i.e. the maximum trading volume allowed in each unit of time.

The market conditions, or the asset price, fluctuate driven by public information and two traders decide when to acquire information on the asset payoff. Motivated by the trade-offs mentioned above, two specific externalities are present: *information externality*, i.e. the spillover of the information from the informed trader to the uninformed trader, and *trading externality*, i.e. an informed trader's trading volume and profitability being affected by the presence of another informed trader.

Once a trader acquires information, some additional information may be disclosed to the other trader. Think of two traders in one asset management firm. If one trader acquires information, then the other trader may observe some evidence from the first trader's trading records, notes, or even from rumors. This possible information spillover is the information externality and, modeled as a signal; it can be observed by the uninformed trader. This assumption makes the second-mover's position advantageous because they will have the opportunity to free-ride the first-mover's information. Therefore, the strength of the second-mover advantage is determined by the accuracy of the signal and the extent to which the uninformed trader can free-ride on the information acquired by the first-mover.

The assumption on trading externality is motivated by the observation that competition on trading can make traders trade less aggressively and reduce their expected profit. In Kyle's trading with multiple informed agents, e.g., Back et al. (2000), the profit of imperfectly competing traders is lower than that of a single informed traders. More importantly, the trading externality assumption strengthens the first-mover advantage because it allows the trader to trade more aggressively to profit from the information if she can acquire the information earlier than the other trader.

Consequently, whether the first-mover advantage dominates the second-mover advantage, or vice versa, determines the nature of the game. If the first-mover advantage dominates, then the timing of the information acquisition is a *preemption game*, where the first trader who acquires information wants to act earlier. Otherwise, if the second-mover advantage dominates, the timing game is a *war of attrition*, where traders will be reluctant to take the lead in information acquisition.

In contrast to the situation where there is only one trader, the trader will acquire information earlier in a preemption game with little information externality. However, when the information externality is strong enough, and hence so is the second-mover advantage, the trader will acquire the information later, even though the timing game is a preemption game. Moreover, the trader will acquire information strictly later than the timing decision when there is only one trader in a war of attrition when the information externality is strong enough.

Recently some papers extend the discussion on information acquisition to the dynamic trading environments, e.g., Mendelson and Tunca (2003), Avdis (2016), but the information acquisition is indeed static–the trader can only decide to acquire information prior to the trading starts. Banerjee and Breon-Drish (2016) are the first to study when a trader chooses to be informed conditional on the public signals. Although there are several papers (e.g., Holden and Subrahmanyam (1992), Foster and Viswanathan (1996), Back et al. (2000)) studying the dynamic trading with imperfect competition among several informed traders, there are no prior studies on the competition on timing of information acquisition among traders. This paper thus fills the gap and motivated to study when and how two traders compete for acquiring information.

Moreover, similar to Banerjee and Breon-Drish (2016), this paper is also based on the real option approach to irreversible investment (e.g. Dixit and Pindyck (1994)). To study the strategic interaction of two traders, our model is built on the framework of real option games. Fudenberg and Tirole (1985) studies the timing game without uncertainty, while Smets (1994) first studies the continuous-time option game in his doctoral dissertation. A number of papers (e.g. Huisman and Kort (1999),Lambrecht (2001), Grenadier (2002), Murto (2004), Dias and Teixeira (2010), etc.) extends the real option game approach in one way or another. In particular, Décamps and Mariotti (2004) and Thijssen et al. (2006) consider the externalities from learning and information which cause second-mover advantage in the timing game. Last but not least, it is very instructive of Dutta and Rustichini (1993) to solve the equilibria of the model in this study.

The plan for the rest of the paper is as follow. Section 2.2 describes the set-up of the model. Then, in Section 2.3, the timing decisions without strategic interaction are studied, which are served as i) benchmark for comparison with the timing decision with strategic interaction and ii) foundations for further analysis of the equilibrium in the timing game. Then we analyze the equilibria of the timing decisions with strategic considerations in Section 2.4. All the proofs are collected in the Appendix.

2.2 The Model

2.2.1 Model Setup

The time $t \in [0, \infty)$ is continuous. Fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ on which the standard Brownian motion (B, Z) and independent random variable v and T are defined.

There is a risky asset trading in the market. The asset will be liquidated at a random date T, where T follows an exponential distribution $\exp(\lambda)$ with rate parameter λ. Following Banerjee and Breon-Drish (2016), assume that the liquidation value of the asset is

$$V = vY_T. \tag{2.2.1}$$

Y is a publicly observable process, which can be interpreted as news or public information, and it evolves stochastically as

$$Y_t = \sigma B_t \tag{2.2.2}$$

with B a standard Brownian motion. However, $v \in \{0, 1\}$ is unobservable unless someone acquires information and reveals it privately to herself. Agents have a common prior belief $\Pr(v = 1) = p \in (0, 1)$ before revealing it or receiving any signal.

The asset price P_t is exogenously determined to be the expectation of its final payoff given public information, i.e.

$$P_t = E_t[V] = pY_t \tag{2.2.3}$$

where $E_t[\cdot] = E[\cdot|(Y_s)_{s\leq t}]$. Note that Y_t is a martingale w.r.t. the filtration generated by (Y_t) so the price is a martingale as well. This assumption can be interpreted as that there is a risk neutral market maker who sets the price according to public information. Moreover, the trading volume from the potentially informed traders is supposed to be far less than the total trading volume in the market, so it has barely any effects on the price.

There are two traders $i \in \{1, 2\}$, who are not informed at time 0, meaning that they believe that $v = 1$ with probability p as public belief. However, each trader i can observe the value of v privately at a fixed cost c at any time $t \geq 0$. We call the trader whoever acquires information first the *leader*, and the other trader the *follower*, if they are not acquiring information simultaneously.

Due to the assumption that the asset price is exogenously determined, once some trader acquires information the other agent cannot learn anything from the price. However, to introduce some *information externality*, we follow the idea similar to Fajgelbaum et al. (forthcoming) such that, whenever someone acquires information first, a free binary signal $s \in \{0, 1\}$ is revealed to the other trader. This assumption captures the effect of social learning in an environment with multiple agents. One trader may observe the other trader directly or hear some rumor so that she can get some clue about the information acquired by the other. Moreover, for simplicity, the signal is assumed to be symmetric in the sense that

$$\Pr(s = 1|v = 1) = \Pr(s = 0|v = 0) = q \in [\frac{1}{2}, 1]. \tag{2.2.4}$$

The value q hence characterizes the accuracy of the signal: the signal is fully revealing the value v if $q = 1$, but it is not uninformative if $q = 1/2$. Typically, we assume $q \in (1/2, 1)$ so that the signal is informative and yet not fulling revealing. However, we also consider extreme cases when either the signal is uninformative, i.e. $q = 1/2$, and it is fully revealing, i.e. $q = 1$ occasionally in the subsequent analysis. This assumption on information externality is crucial to introduces second-mover advantage in the timing game. With possibility of free-riding the information through the signal from the leader, it gives agents the incentive to become a follower.

Let us denote ρ^s the unconditional probability of signal s. Then,

$$\rho^1 = pq + (1-p)(1-q), \tag{2.2.5}$$

$$\rho^0 = p(1-q) + (1-p)q. \tag{2.2.6}$$

Moreover, we denote $p^s = \Pr(v = 1|s)$ the posterior of the follower given signal s, and by Bayes rule,

$$p^1 = \frac{pq}{\rho^1} = \frac{pq}{pq + (1-p)(1-q)}; \tag{2.2.7}$$

$$p^0 = \frac{p(1-q)}{\rho^0} = \frac{p(1-q)}{p(1-q) + (1-p)q}. \tag{2.2.8}$$

Besides the information externalities through the free signal, the information acquisition affects the *maximum trading volume*, or the *trading capacity*, of each trader and her opponent in different situations. Without acquiring information, each trader can trade, i.e. buy or sell, at most θ_0 shares in each unit of time. Moreover, information acquisition is accompanied by increased trading capacity. If trader i, the leader, acquire the information first, then she can trade at most $\theta_1 = \hat{\theta}_1 + \theta_0$ shares, and the other trader $-i$, the follower, can still trade θ_0 share in each unit of time when she has not acquired information. This assumption captures that whenever someone has advantage in information, she will trade more aggressively. Think of that, in a asset management company, if one trader believes that she has superior information, then she will persuade the fund manager to allocation more capital to her trading account.

Furthermore, if both traders acquire information, each trader i can trade at most $\theta_2 = \hat{\theta}_2 + \theta_0$ share in each unit of time; and we assume that $\hat{\theta}_1 > \hat{\theta}_2 \geq 0$ so that we have $\theta_1 > \theta_2 \geq \theta_0$. This means that the trading capacity of the leader will be reduced from θ_1 to θ_2, while the follower's trading capacity will increase from θ_0 to θ_2, when the follower acquires information. It is essentially saying that, when the follower acquires information, she will trader more aggressively and squeeze out the leader in the market. Again, think of that, if two traders both have the information and come to the fund manager, the fund manager will increase the capital allocation to each trader; but there is an upper limit for the total capital available for these two traders, so the fund manager cannot increase capital allocation to each trader as much as that for only one informed trader. This assumption captures the externality from competition: when another trader, the follower, acquires information, she act as an entrant to the market and encroach the profit of the incumbent or the leader in the market, the trader who acquires information first.

Let τ_i denote the time of trader i acquiring information and x_{it} denote the share of the asset purchased (or sold if it is negative) by trader i at time t. Then, we assume that traders are risk neutral, and they maximize their expected profit from trading, i.e. their objective function is

$$E\left[\int_0^T (V - P_u) x_{iu} du\right] = E\left[\int_0^\infty e^{-\lambda u}(V - P_u) x_{iu} du\right] \tag{2.2.9}$$

where the equality holds because $T \sim Exp(\lambda)$. However, the explicit expression of the expected profit depends on who moves first, for example, if $\tau_i \leq \tau_j$, i.e. trader i

is the leader and trader j is the follower, then the expected profit for trader i and j, respectively, are

$$\Pi_i = E\left[\int_0^{\tau_i} e^{-\lambda u}(V - P_u)x_{iu}du + E\left[\int_{\tau_i}^{\tau_j} e^{-\lambda u}(V - P_u)x_{iu}du + \int_{\tau_j}^{\infty} e^{-\lambda u}(V - P_u)x_{iu}du|v\right]\right], \tag{2.2.10}$$

$$\Pi_j = E\left[\int_0^{\tau_i} e^{-\lambda u}(V - P_u)x_{ju}du + E\left[\int_{\tau_i}^{\tau_j} e^{-\lambda u}(V - P_u)x_{ju}du|s\right] + E\left[\int_{\tau_j}^{\infty} e^{-\lambda u}(V - P_u)x_{ju}du|v\right]\right]. \tag{2.2.11}$$

Note that during the time period between τ_i and τ_j, two traders have different information and different trading capacity.

2.2.2 Trading Decision and Expected Profit Flow

In this subsection, we fix the timing decision τ_i for each trader and then analyze the trading decision only and compute the expected profit flow at each unit of time. Consider the trading decision for each trader in three scenarios: 1) Both traders have not acquired information; 2) one trader has acquired information, but the other has not; and 3) both traders have acquired information. And in the second scenario, we discuss trading decisions for both a leader and a follower. Since traders are *ex ante* identical and role of the trader, either a leader or a follower, captures the difference of the traders, the subscripts i are dropped in this subsection and focus on the trading decision for different roles of the traders in different scenarios. Much of the analysis on timing of information acquisition resembles the analysis of timing of entering market in the industrial organization literature. From this perspective, the second scenario is called the *information monopoly phase* because it is like the trader who has acquired information has monopoly power over the true valuation. Once the follower acquires information, then the competition between two traders resembles the competition between two firms in a duopoly market, and the third scenario is called the *information duopoly phase.*

Because there is no strategic or intertemporal effects for individual trading activities, the optimal trading problem can be considered as a series of static individual optimization problems, i.e. optimal trading x_t simply maximizes the flow of expected profit, i.e.

$$x_t \in \arg\max E\left[(V - P_t)x|I_t\right] \tag{2.2.12}$$

where $E[\cdot|I_t]$ is the expectation operator with respect of trader's information set I_t at time t. Note that, for $t < T$,

$$\begin{aligned} E_t\left[(V - P_t)x_t\right] &= E_t\left[(vY_T - pY_t)x_t\right] \\ &= Y_t\left(E_t\left[v\right] - p\right)x_t, \end{aligned} \tag{2.2.13}$$

where the second equality holds because Y_t is a martingale. Therefore, the trading decision is simply comparing the valuation of the trader with the public belief.

Suppose the trader has not acquired information. She holds the same belief as market maker so that the asset price is fair to her, i.e. $E_t\left[v\right] = p$, and she is indifferent to buy or sell the asset. To pin down their trading decision uniquely, impose that $x_t = 0$ if the trader has not acquired information. This assumption is innocuous because the expected profit will be zero no matter what x_t is.

In the information monopoly phase, if one has acquired information but the other has not, then the additional information they see generates difference between their belief and public belief. And if $E_t\left[v\right] \neq p$, the solution must be bang-bang because of the objective function is linear in x_t. Therefore, given that $p \in (0, 1)$, the optimal trading decision for the leader, who has acquired information, is as follow:

$$x_t = \begin{cases} \theta_1 & v = 1 \\ -\theta_1 & v = 0 \end{cases}. \tag{2.2.14}$$

The follower, who has not acquired information but received the signal s, optimally trades as follow:

$$x_t = \begin{cases} \theta_0 & s = 1 \\ -\theta_0 & s = 0 \end{cases}. \tag{2.2.15}$$

Finally, when both traders have revealed v, then they do not have asymmetry in information and difference in trading capacity any more and both choose

$$x_t = \begin{cases} \theta_2 & v = 1 \\ -\theta_2 & v = 0 \end{cases}. \tag{2.2.16}$$

Given the analysis of optimal trading decisions in different scenarios, the expected profit flow in each unit of time can be computed under different situations easily. However, first define the following *profit multipliers*: for $v = 0, 1$ and $s = 0, 1$,

$$\pi_{L,v} = |v - p|\theta_1, \tag{2.2.17}$$

$$\pi_F^s = |p^s - p|\theta_0 = (2q-1)p(1-p)\theta_0/\rho^s, \tag{2.2.18}$$

$$\pi_{D,v} = |v - p|\theta_2, \tag{2.2.19}$$

where $\pi_{L,v}$ and π_F^s are the profit multipliers for the leader who reveals v and the follower who receives signal s, respectively, in the information monopoly phase, and $\pi_{D,v}$ is the profit multiplier for both traders conditional on v in the information duopoly phase. The, for example, the leader who reveals $v = 1$ in the information monopoly phase has expected profit flow

$$\begin{aligned} E_t\left[(V-P_t)x_t\right] &= Y_t\left(E_t\left[v\right]-p\right)\theta_1 \\ &= Y_t(1-p)\theta_1 = Y_t\pi_{L,1} \end{aligned} \tag{2.2.20}$$

where the first equality follow Equation (2.2.13) and optimal trading decision for the leader (2.2.14), and the second equality due to the fact that the in the leader's information set $v = 1$, and, finally, Equation (2.2.17) implies that $\pi_{L,1} = (1-p)\theta_1$. Therefore, using the similar arguments, the following list summarizes expected profit flows in some situations:

- $Y_t\pi_{L,v}$ —the expected profit flows of the leader who reveals information v in the information monopoly phase,
- $Y_t\pi_F^s$ —the expected profit flow of the follower who receives signal s in the information monopoly phase, and
- $Y_t\pi_{D,v}$—the expected profit flow for both traders conditional on v in the information duopoly phase.

In the same fashion, let us define some other useful profit multipliers as follow. The unconditional expected profit multiplier π_L of a leader is

$$\pi_L = p\pi_{L,1} + (1-p)\pi_{L,0} = 2p(1-p)\theta_1; \tag{2.2.21}$$

the unconditional expected profit multiplier π_F of a follower is

$$\pi_F = \rho^1\pi_{F,1} + \rho^0\pi_{F,0} = 2p(1-p)(2q-1)\theta_0; \tag{2.2.22}$$

the unconditional expected profit multiplier π_D of a trader in the information duopoly phase is

$$\pi_D = p\pi_{D,1} + (1-p)\pi_{D,0} = 2p(1-p)\theta_2. \tag{2.2.23}$$

Moreover, the profit multiplier π_D^s for the information duopoly phase of a follower receiving signal s is

$$\pi_D^s = p^s\pi_{D,1} + (1-p^s)\pi_{D,0} = p(1-p)\theta_2/\rho^s. \tag{2.2.24}$$

Correspondingly,

- $Y_t\pi_L$ and $Y_t\pi_F$ are the unconditional expected profit of a leader and a follower, respectively,
- $Y_t\pi_D$ is the unconditional expected profit flow during the information duopoly phase, and
- $Y_t\pi_D^s$ is the expected profit flow in the information duopoly phase of a follower who just knows the signal s.

2.3 The Timing Decision without Strategic Interaction

This section studies the timing decision without strategic consideration in four scenarios: 1) the timing decision when there is only one trader who can acquire information; 2) the timing decision of a designated follower; 3) the timing decision of a designated leader; and 4) the optimal timing of acquiring information when two traders have to move simultaneously. Since in each of the scenarios the role of the traders are designated, the subscriptions i are dropped in this section. The purpose of this section is, first, to set a benchmark to contrast them with the situation when there are strategic interactions and, second, to serve as a starting point to analyze the strategic timing decisions.

2.3.1 Information Monopoly

First, to set a benchmark, assume there is only one trader who can potentially acquire information in this subsection, and we refer her the *information monopoly* once she acquires information. The environment, including the information structure and trading arrangement, are the same as before, except that, since there is only one trader can acquire information, it is not necessary to have the information spillover, the free signal generated by acquiring information, and the trader has no need to concern the competition in the future to decrease her trading capacity from θ_1 to θ_2.

Then the problem becomes a standard individual stopping problem. Suppose that the trader decides whether or not to acquire information at time $\tau < T$. At this point that she has not revealed v, her unconditional expected profit flow is $Y_\tau \pi_L$ in each unit of time from time τ on, the present value of the total expected profit is $\int_0^\infty e^{-\lambda u} Y_\tau \pi_L du = Y_\tau \pi_L / \lambda$. Note that there is a cost c of acquiring information, hence, like a perpetual American put option, the final payoff at the time τ of acquiring information is $(Y_\tau \pi_L/\lambda - c)^+$ where $(x)^+ = \max\{0, x\}$. The optimal timing τ solve the following problem:

$$M(y) = \max_{\tau} E\left[e^{-\lambda\tau}\left(\frac{Y_\tau \pi_L}{\lambda} - c\right)^+ |Y_0 = y\right] \tag{2.3.1}$$

where $M(y)$ is the *value function* for the trader when the current state, i.e. the public information, is $Y_0 = y$.

By standard textbook argument (see e.g., Dixit and Pindyck (1994)), the optimal timing has the form

$$\tau_M = \inf\{t \geq 0 | Y_t \geq Y_M\} \tag{2.3.2}$$

for some threshold $Y_M > 0$. Denote a constant β, which will be useful throughout the paper, as a positive root for the equation $\lambda = \frac{1}{2}\sigma^2\beta(\beta-1)$,

$$\beta = \frac{1+\sqrt{1+8\lambda/\sigma^2}}{2}. \tag{2.3.3}$$

Then, the solution to the problem is solved and summarized in the following proposition.

Proposition 2.3.1. *The optimal timing solution to the Problem (2.3.1) is* $\tau_M = \inf\{t \geq 0 | Y_t \geq Y_M\}$ *with*

$$Y_M = \frac{\beta}{\beta-1}\frac{\lambda c}{\pi_L}. \tag{2.3.4}$$

The value function is

$$M(y) = \begin{cases} \left(\frac{y}{Y_M}\right)^\beta \frac{Y_M \pi_L}{\beta\lambda} & y < Y_M, \\ \frac{y\pi_L}{\lambda} - c & y \geq Y_M. \end{cases} \tag{2.3.5}$$

2.3.2 The Follower's Problem

Next, think of the environment with two potentially informed traders. When one trader has acquired information, the other trader, the follower who sees the signal s, choose her timing τ_F^s to acquire information. This problem is another individual stopping problem. However, the follower, depending on the signal s, has some information advantage over the market maker, so she trades and has positive expected profit flow $Y_t\pi_F^s$ even before the she acquires information. After she acquires information at time τ_F^s, the expected profit flow would be $Y_t\pi_{D,v}$ since it will enter an information duopoly phase when both traders acquire information. Hence, the expected profit of a follower who acquires information at τ_F^s conditional on current state $Y_0 = y$ is

$$\begin{aligned} J^s[\tau_F^s; y] &= E\left[\int_0^{\tau_F^s} e^{-\lambda u} Y_u \pi_F^s du + \left(\int_{\tau_F^s}^{\infty} e^{-\lambda u} Y_u \pi_{D,v} - e^{-\lambda \tau_F} c\right)^+ du | s, Y_0 = y\right] \\ &= E\left[\int_0^{\tau_F^s} e^{-\lambda u} y \pi_F^s du + e^{-\lambda \tau_F^s} \left(\frac{y \pi_D^s}{\lambda} - c\right)^+ du | s, Y_0 = y\right] \end{aligned} \tag{2.3.6}$$

for $s \in \{0, 1\}$. Hence, her problem is

$$F^s(y) = \max_{\tau_F^s} J^s[\tau_F^s; y] \tag{2.3.7}$$

where $F^s(y)$ is the value function for the designated follower who receives signal s. Again, consider the optimal timing has the form $\tau_F^s = \inf\{t \geq 0 | Y_t \geq Y_F^s\}$ with threshold Y_F^s.

Proposition 2.3.2. *Suppose that $\hat{\theta}_2 > 0$ or $q < 1$. Then the solution to the optimal stopping problem (2.3.7) is $\tau_F^s = \inf\{t \geq 0 | Y_t \geq Y_F^s\}$ with*

$$Y_F^s = \frac{\beta}{\beta - 1} \frac{\lambda c}{\pi_D^s - \pi_F^s}, \tag{2.3.8}$$

and the value function is

$$F^s(y) = \begin{cases} \frac{Y_F^s(\pi_D^s - \pi_F^s)}{\beta \lambda} \left(\frac{y}{Y_F^s}\right)^\beta + \frac{y \pi_F^s}{\lambda} & y < Y_F^s, \\ \frac{y \pi_D^s}{\lambda} - c & y \geq Y_F^s, \end{cases} \tag{2.3.9}$$

where β is defined by equation (2.3.3). Moreover, if either (i) $\hat{\theta}_2 = 0$ and $q \to 1$ or (ii) $\hat{\theta}_2 \to 0$ and $q = 1$, then $Y_F^s \to \infty$ and $\tau_F^s \to \infty$.

The optimal timing problem for each type of follower is solved in the same way as that in the last subsection. However, the last statement of the proposition means that, if acquiring information is not improving the follower's information advantage or trading capacity, then the follower will never acquire information by herself. So, to make the timing problem for the follower meaningful, $\hat{\theta}_2 > 0$ or $q < 1$ is usually imposed if there is no other clarification.

It is also useful to compute the *ex ante* expected value $F(y)$ to be a follower because the trader has not seen the signal when later it comes to the analysis whether a trader choose to become a leader or a follower. Simply, we take unconditional expectation of the follower's value:

$$
\begin{aligned}
F(y) &= \rho^0 F^0(y) + \rho^1 F^1(y) \\
&= \begin{cases}
\frac{\hat{Y}_F}{\beta}\frac{\pi_D-\pi_F}{\lambda}\left(\frac{y}{\hat{Y}_F}\right)^\beta + \frac{y\pi_F}{\lambda} & y < \min\{Y_F^0, Y_F^1\} \\
\frac{Y_F^0}{\beta}\frac{\pi_D-\pi_F}{2\lambda}\left(\frac{y}{Y_F^0}\right)^\beta + \frac{y(\pi_D+\pi_F)}{2\lambda} - \rho^1 c & Y_F^1 \le y < Y_F^0 \\
\frac{Y_F^1}{\beta}\frac{\pi_D-\pi_F}{2\lambda}\left(\frac{y}{Y_F^1}\right)^\beta + \frac{y(\pi_D+\pi_F)}{2\lambda} - \rho^0 c & Y_F^0 \le y < Y_F^1 \\
\frac{y\pi_D}{\lambda} - c & y \ge \max\{Y_F^1, Y_F^0\}
\end{cases}
\end{aligned}
\tag{2.3.10}
$$

where

$$
\hat{Y}_F^{1-\beta} = \frac{1}{2}\left[(Y_F^1)^{1-\beta} + (Y_{F,0}^0)^{1-\beta}\right]. \tag{2.3.11}
$$

Note that $\hat{Y}_F$ is a power mean of Y_F^1 and Y_F^0 with exponent $1-\beta$ so that it must be that $\min\{Y_F^0, Y_F^1\} \le \hat{Y}_F \le \max\{Y_F^0, Y_F^1\}$.

The change in signal, q, as a parameter for the value function provides interesting comparative statics. Denote $F(y;q)$ the value function of the designated follower with $Y_0 = y$ and given the signal accuracy q. Then, intuitively, the follower can grasp more benefit from it in expectation with the accuracy of signal increased. Therefore, if q is higher, $F(y;q)$ will be also higher for each $y < \max\{Y_F^1, Y_F^0\}$.

Lemma 2.3.1. *$F(y;q)$ is strictly increasing in $q \in [1/2, 1]$ for each $y < \max\{Y_F^0, Y_F^1\}$, and it is constant w.r.t. q when $y \ge \max\{Y_F^0, Y_F^1\}$.*

Moreover, the thresholds Y_F^s's also change w.r.t. the change in signal precision q.

Lemma 2.3.2. *If $1/2 < p < 1$, then $Y_F^1 \geq Y_F^0$, moreover, Y_F^1 is increasing in q and Y_F^0 is decreasing in q.*

If $0 < p < 1/2$, then $Y_F^0 \geq Y_F^1$, moreover, Y_F^0 is increasing in q and Y_F^1 is decreasing in q.

If $p = 1/2$, then $Y_F^1 = Y_F^0$, moreover, both Y_F^0 and Y_F^1 is increasing in q.

As a corollary of Lemma 2.3.2, $\max\{Y_F^0, Y_F^1\}$ is increasing in the precision q of the signal.

2.3.3 Leader's Value and Designated Leader's Problem

Now consider another situation with two traders with their roles predetermined: one trader, the designated follower, has to acquire information after the other trader, the designated leader. The designated follower will choose her timing optimally just as specified in Section 2.3.2. To formulate and solve the designated leader's problem, the first thing to do is to compute the *leader's value $L(y)$ during the informational monopoly phase.*

Given that the leader observes v and generates signal s, and the other player will always be a follower and act optimally, then the following L_v^s is the value function for such a leader

$$\begin{aligned} L_v^s(y) &= E\left[\int_0^{\tau_F^s} e^{-\lambda u}(V - P_u)x_u du + \int_{\tau_F^s}^{\infty} e^{-\lambda u}(V - P_u)x_u du | v, s, Y_0 = y\right] \\ &= E\left[\int_0^{\tau_F^s} e^{-\lambda u} Y_u \pi_{L,v} du + \int_{\tau_F^s}^{\infty} e^{-\lambda u} Y_u \pi_{D,v} du | Y_0 = y\right] \\ &= \begin{cases} \frac{y\pi_{L,v}}{\lambda} + \left(\frac{y}{Y_F^s}\right)^{\beta} \frac{Y_F^s\left(\pi_{D,v} - \pi_{L,v}\right)}{\lambda} & y < Y_F^s \\ \frac{y\pi_{D,v}}{\lambda} & y \geq Y_F^s \end{cases}. \end{aligned} \tag{2.3.12}$$

To obtain the leader's value L, take the expectation over v and s:

$$
\begin{aligned}
L(y) &= \sum_{v,s} \Pr(v,s) L_v^s - c \\
&= \begin{cases}
\frac{y\pi_L}{\lambda} + \left(\frac{y}{\hat{Y}_F}\right)^{\beta} \frac{\hat{Y}_F(\pi_D - \pi_L)}{\lambda} & y < \min\left\{Y_F^0, Y_F^1\right\} \\
\frac{y(\pi_D + \pi_L)}{2\lambda} + \left(\frac{y}{Y_F^0}\right)^{\beta} \frac{Y_F^0(\pi_D - \pi_L)}{2\lambda} & Y_F^1 \le y < Y_F^0 \\
\frac{y(\pi_D + \pi_L)}{2\lambda} + \left(\frac{y}{Y_F^1}\right)^{\beta} \frac{Y_F^1(\pi_D - \pi_L)}{2\lambda} & Y_F^0 \le y < Y_F^1 \\
\frac{y\pi_D}{\lambda} & y \ge \max\left\{Y_F^0, Y_F^1\right\}
\end{cases}
\end{aligned} \tag{2.3.13}
$$

Then, the designated leader's problem is

$$
V(y) = \max_{\tau} E\left[e^{-\lambda\tau} \left(L(Y_\tau)\right)^{+} | Y_0 = y\right] \tag{2.3.14}
$$

where $V(y)$ is the value function of a designated leader when current state, i.e. the public information, is y. This optimal stopping problem can be solved with similar method as before in the Appendix, except that we need to make additional efforts to discuss different cases.

Proposition 2.3.3. *The solution to the optimal stopping problem (2.3.14) is*

$$
\tau_L = \inf\left\{t | Y_t \ge Y_L\right\} \tag{2.3.15}
$$

where

$$
Y_L = \begin{cases}
\frac{\beta}{\beta - 1} \frac{\lambda c}{\pi_L} & \pi_L > \max_s \left\{\pi_D^s - \pi_F^s\right\}, \\
\frac{\beta}{\beta - 1} \frac{2\lambda c}{\pi_L + \pi_D} & \pi_L \le \max_s \left\{\pi_D^s - \pi_F^s\right\}.
\end{cases} \tag{2.3.16}
$$

2.3.4 Simultaneous Move

For the last scenario without strategic interaction, consider the case that two traders acquire information always at the same time. This means that no one has the opportunity to see the signal to make decision, moreover, they directly enter the information duopoly phase after acquiring information. Hence, their optimal stopping problem is just like a standard individual stopping problem, but their expected profit flow would be $Y_t\pi_D$ when they acquire information, i.e.

$$S(y) = \max_{\tau} E\left[e^{-\lambda\tau}\left(\frac{Y_\tau\pi_D}{\lambda} - c\right)^+ |Y_0 = y\right] \tag{2.3.17}$$

where $S(y)$ is the *value function* for the trader when the current state is y.

Proposition 2.3.4. *The solution to the Problem (2.3.17) is $\tau_S = \{t \geq 0 | Y_t \geq Y_S\}$ with*

$$Y_S = \frac{\beta}{\beta - 1}\frac{\lambda c}{\pi_D}, \tag{2.3.18}$$

and the value function S is as follow:

$$S(y) = \begin{cases} \left(\frac{Y_S\pi_D}{\lambda} - c\right)\left(\frac{y}{Y_S}\right)^\beta & y < Y_S, \\ \frac{y\pi_D}{\lambda} - c & y \geq Y_S. \end{cases} \tag{2.3.19}$$

2.4 Equilibrium Analysis

In this section, the strategic interaction of two traders is analyzed by the approach of real option games. The equilibrium notion employed here is the *Markov perfect equilibrium.* In fact, we focus on a specific class of MPE, namely the *stopping equilibria* as defined by Dutta and Rustichini (1993).

Note that once the first trader acquires information and becomes the leader, the other remaining trader receives the signal s and get the terminal payoff as a follower, whose value and optimal choice are fully characterizes in the follower's problem of the Section 2.3.2. Therefore our attention can be restricted to a game that ends when one of the traders acquires information. Consequently, it is sufficient to only consider the the timing strategy τ_i of each trader i when no one has acquired information, and the equilibrium (τ_1^*, τ_2^*) of the game would be a pair of such timing strategies for each trader. In particular, we focus on a specific class of Markov strategies as considered in Dutta and Rustichini (1993), that is, the traders i choose the exercise boundary Y_i of the stopping time $\tau_i = \inf\{t \geq 0 | Y_t \geq Y_i\}$ when no one has acquired information. Let's denote $\mathcal{T}$ denote the set of all such strategies.

An other issue associated with a timing game is that players may choose to stop at the same time. To deal with this, the usual assumption is adopted here: If two traders move at the same time, then with one-half probability, one of them get the information and the other could not, and the roles of two traders switch otherwise. The trader who fails to acquire information has a second chance to choose to either become a follower, receiving the signal and then timing optimally, or try immediately again without looking at the signal. Two traders move simultaneously as considered in the Section 2.3.4 if it is the second case. More specifically, suppose at time t both traders attempt to acquire information. The one who fails to reveal the information will choose terminal payoff $\max\{F(Y_t), S(Y_t)\}$ rationally. So only when $\max\{F(Y_t), S(Y_t)\} = S(Y_t)$, the remaining uninformed trader would like to move simultaneously, i.e. try again immediately without revealing the signal.

However, the next lemma shows that it is never optimal to move simultaneously when $q > 1/2$. The intuition is simple: The trader prefer becoming a follower rather than moving simultaneously because of the possibility of free-ride on the information. And in fact, move simultaneously is weekly dominated by being a follower, and strictly dominated when $Y_t < \max\{Y_F^0, Y_F^1\}$.

Lemma 2.4.1. *$F(y) \geq S(y)$ for all $y < \max\{Y_F^0, Y_F^1\}$, and the inequality is strict if $q > \frac{1}{2}$.*

Given the above lemma that the traders never strictly prefer to move simultaneously, therefore their payoff can be written as a function of their strategies (τ_1, τ_2)

and the current state y: The payoff of trader i is

$$U_i(\tau_i, \tau_{-i}; y) = E\left[e^{-\lambda\tilde{\tau}}\left\{L(Y_{\tau_i})\chi_{\{\tau_i<\tau_{-i}\}} + F(Y_{\tau_{-i}})\chi_{\{\tau_i>\tau_{-i}\}}\right.\right.$$
$$\left.\left. + \frac{[L(Y_{\tau_i}) + F(Y_{\tau_i})]}{2}\chi_{\{\tau_i=\tau_{-i}\}}\right\}|Y_0 = y\right]$$

where we use notation i and $-i$ in the usual way, if $i = 1$ then $-i = 2$; and vice versa. Then, the equilibrium is defined as usual.

Definition 2.4.1. The strategy profile (τ_1^*, τ_2^*) is an equilibrium if

$$\tau_i^* \in \arg\max_{\tau_i \in \mathcal{T}} U_i(\tau_i, \tau_{-i}^*; y)$$

for $i = 1, 2$ and all $y \geq 0$.

Moreover, there is a distinction between two classes of a real option game or a timing game: a preemption game and a war of attrition. Typically, this two types of timing game prescribe different incentive for the agents to act as early as possible or as late as possible.

Definition 2.4.2. The timing game is a *preemption game* if $L(Y_L) > F(Y_L)$ and it is a *war of attrition* if $L(Y_L) \leq F(Y_L)$.

Specifically, in a preemption game, the agents want to preempt in the following sense: Think of a timing game with two agents and one agents is designated follower, then according to the analysis in the previous section, the designated leader should optimally move at τ_L, i.e. when Y_t reaches the threshold Y_L. However, the designated follower has incentive to deviate: Suppose now the state $Y_t = Y_L - \epsilon$ for some $\epsilon > 0$ small enough so that $L(Y_L - \epsilon) > F(Y_L - \epsilon)$ by continuity, the designated follower would move at this point because she can get value $L(Y_L - \epsilon)$ by preempting instead of getting the follower's value $F(Y_L - \epsilon)$ if she waits for the designated leader moving first. In a war of attrition, on the contrary, the designated follower has no incentive to lead, and in fact the designated leader wants to postpone her move as late as possible.

Moreover, in this timing game, there are two major trade-offs: the leader can enjoy the first-mover advantage, the extra profit during the information monopoly phase by trading more aggressively due to the higher trading capacity, while the follower has the second-mover advantage, the possibility of free-riding on the information acquired by the leader. If the first-mover advantage dominates, then the game is a preemption game, otherwise it is a war of attrition. Therefore, the trading capacity and the accuracy of the signal can alter the nature of this timing games. The following series of lemmas shows how the nature of the game of information acquisition is associated with these parameters.

Lemma 2.4.2. *Suppose $q = 1/2$, then $Y_L = Y_M$ and $L(Y_L) > F(Y_L)$ given that $\hat{\theta}_1 > \hat{\theta}_2 \geq 0$.*

Lemma 2.4.2 says that, if the signal is uninformative, then the game must be a preemption as long as the information monopoly has extended trading capacity because there is only first-mover advantage but no second-mover advantage. Briefly, suppose that in an extreme case that $\hat{\theta}_1 = \hat{\theta}_2 > 0$ and $q = 1/2$, it will turn out to be the case that the equilibrium to this case is exactly to the solution of an individual optimal stopping problem. Moreover, one corollary to this lemma is that the for any $\hat{\theta}_1 > \hat{\theta}_2 \geq 0$, if $q > 1/2$ is small enough, by continuity of leader's value L and follower's value F, it must be the case that $L(Y_L) > F(Y_L)$, and hence, the game is a preemption game.

Proposition 2.4.1. *Given that $\hat{\theta}_1 > \hat{\theta}_2 \geq 0$, the timing game is a preemption game if $1/2 \leq q < q_*$ for some $q_* > 1/2$.*

The above shows that the first-mover advantage dominates when the information externality is possibly weakest, i.e. $q = 1/2$. However, when the information externality is possibly strongest, i.e. $q = 1$, it does not guarantee the second-mover advantage dominate. Some additional conditions are necessary. First define the following functions:

$$h(a,b) = \frac{\beta}{\beta-1}a + \left(\frac{b}{2}\right)^{\beta}\left[p^{1-\beta} + (1-p)^{1-\beta}\right]\left(1 - \frac{\beta}{\beta-1}\frac{a}{b}\right);$$
$$g_0(a,b) = \frac{\beta}{\beta-1}\frac{a}{2(1-p)} + \left(\frac{b}{2(1-p)}\right)^{\beta}\left(1 - \frac{\beta}{\beta-1}\frac{a}{b}\right);$$
$$g_1(a,b) = \frac{\beta}{\beta-1}\frac{a}{2p} + \left(\frac{b}{2p}\right)^{\beta}\left(1 - \frac{\beta}{\beta-1}\frac{a}{b}\right).$$

Lemma 2.4.3. *$L(Y_L) < F(Y_L)$ when $q = 1$ if one of the following conditions hold:*

Condition A: $\pi_L > \max_s \{\pi_D^s - \pi_F^s\}$ and

$$h\left(\hat{\theta}_1/\theta_1, \hat{\theta}_2/\theta_1\right) < 1;$$

Condition B: $\pi_L \leq \pi_D^1 - \pi_F^1$ and

$$g_0\left(\frac{\hat{\theta}_1}{(\theta_1+\theta_2)/2}, \frac{\hat{\theta}_2}{(\theta_1+\theta_2)/2}\right) < 1;$$

Condition C: $\pi_L \leq \pi_D^0 - \pi_F^0$ and

$$g_1\left(\frac{\hat{\theta}_1}{(\theta_1+\theta_2)/2}, \frac{\hat{\theta}_2}{(\theta_1+\theta_2)/2}\right) < 1.$$

Then, by the fact that $L(Y_L) - F(Y_L)$ continuously depends on the parameter q and the fact that $L(Y_L) - F(Y_L) > 0$ when $q = 1/2$, there must exist some $1/2 < q^* < 1$ such that $L(Y_L) - F(Y_L) < 0$ for all $q > q^*$. Hence, the following proposition summarizes the conditions when the timing game is a war of attrition.

Proposition 2.4.2. *Under either Condition A, B, or C, there exists some $q^* \in (1/2, 1)$ such that the timing game is a war of attrition if $q \geq q^*$.*

We say that the first-mover advantage is weak if $h\left(\hat{\theta}_1/\theta_1, \hat{\theta}_2/\theta_1\right)$, $g_0\left(\frac{\hat{\theta}_1}{(\theta_1+\theta_2)/2}, \frac{\hat{\theta}_2}{(\theta_1+\theta_2)/2}\right)$, and $g_1\left(\frac{\hat{\theta}_1}{(\theta_1+\theta_2)/2}, \frac{\hat{\theta}_2}{(\theta_1+\theta_2)/2}\right)$ is small because they are equivalent to that $L(Y_L)-F(Y_L)$ is small. Moreover, these functions, $h(a,b)$, $g_0(a,b)$, and $g_1(a,b)$, are all increasing in the first argument a and decreasing in the second argument b when $b \leq a$. Note that increase in $\hat{\theta}_1$, which is essentially strengthening the first-mover advantage, will increase $\hat{\theta}_1/\theta_1$ and $\frac{\hat{\theta}_1}{(\theta_1+\theta_2)/2}$ and decrease $\frac{\hat{\theta}_2}{(\theta_1+\theta_2)/2}$. But increase in $\hat{\theta}_2$ will increase $\hat{\theta}_2/\theta_1$ and $\frac{\hat{\theta}_2}{(\theta_1+\theta_2)/2}$ but decrease $\frac{\hat{\theta}_1}{(\theta_1+\theta_2)/2}$, hence, the first-mover will be weaken. Therefore, the interpretation of Conditions A, B, and C are essentially the same: it requires that the trading capacity, both $\hat{\theta}_1$ and $\hat{\theta}_2$, of informed traders to be sufficiently small, especially a small $\hat{\theta}_2$, will render this timing game of information acquisition a war of attrition along with a strong enough second-mover advantage or a high enough accuracy of the signal.

After discussing when the timing game of information acquisition is a preemption game and when it is a war of attrition, the equilibrium is analyzed in these two different cases. First, assume that the timing game is a preemption game, that is $L(Y_L) > F(Y_L)$. If one trader, say trader 1, believes that the other trader, trader 2, will not preempt, then it is optimal to acquire information when Y_t reaches Y_L. Suppose that $\epsilon > 0$ is sufficiently small, then $L(Y_L - \epsilon) > F(Y_L - \epsilon)$ by continuity. However, at this time, by preempting at $Y_L - \epsilon$, trader 2's expected utility jumps from $F(Y_L - \epsilon)$ to $L(Y_L - \epsilon)$ so that she is strictly better off. Applying the same logic, then trader 1 will want to preempt when $Y_t = Y_L - 2\epsilon$ if she knows that trader 2 acquires information at $Y_L - \epsilon$, and so on and so forth. This process of one preempting another continues until it reaches the point such that $L(Y_P) = F(Y_P)$ for some $Y_P < Y_L$; and the following lemma established the existence and uniqueness of such Y_P.

Lemma 2.4.4. *Suppose that $L(Y_L) > F(Y_L)$. Then there exists a unique $Y_P \in (0, Y_L)$ such that $L(Y_L) = F(Y_L)$.*

Moreover, since two traders are symmetric, the above argument can be symmetrically applied to both traders, so the only possible equilibrium is that both traders want to acquire information at Y_P. We can verify that it is an equilibrium and it is unique.

Proposition 2.4.3. *Suppose that the timing game is a preemption game, i.e.* $L(Y_L) > F(Y_L)$*. Then there exists a unique equilibrium* (τ_1^*, τ_2^*) *such that*

$$\tau_1^*(y) = \tau_2^*(y) = \inf \{t \geq 0 : Y_t \geq Y_P\}.$$

To contrast the timing decision of an information monopoly and a leader's timing in an equilibrium of a preemption game, simply compare their thresholds in equilibrium. A natural guess is that the leader in a preemption game acts earlier than the information monopoly because traders want to preempt. However, it is not quite correct. In general, if there is no information externality, then $Y_L = Y_M$, and the preemption game requires the leader's threshold $Y_P < Y_L$, hence, $Y_P < Y_M$ in equilibrium. But it is only partially true when there is information externality, in particular when $\pi_L > \max_s \{\pi_D^s - \pi_F^s\}$, which implies that $Y_M = Y_L = Y_P$. Otherwise, we only know that $Y_P < Y_L$, but since $Y_M < Y_L$ when $\pi_L \leq \max_s \{\pi_D^s - \pi_{F,s}\}$, it is entirely possible to have $Y_M < Y_P < Y_L$ for some cases. Therefore the presence of information externality causes competing traders delay their information acquisition possibly because of the second-mover advantage from the information spillover.

Then consider the case where the timing game of information acquisition is a war of attrition, i.e. $L(Y_L) \leq F(Y_L)$. In the case of a war of attrition, the second-mover advantage dominates, therefore both traders want to be a follower. Similar to a standard war of attrition, there can be two pure strategy equilibria: one concedes at the beginning of the war and the other insists to the end with different roles of players in different equilibria. In this timing game, two equilibria are in the similar fashion, except that the leader concedes or acquires information not at the beginning of the game, but at the first time that is optimal to be a leader, which the other guy who would become the follower and insists until it is not optimal to wait anymore. Specifically, let us again denote $Y_P < \max\{Y_F^0, Y_F^1\}$ to be a point such that $L(Y_P) = F(Y_P)$ if it exist, and denote $Y_* = \min\{Y_P, \max\{Y_F^0, Y_F^1\}\}$.[1]

Proposition 2.4.4. *Suppose that the timing game is a war of attrition, i.e.* $L(Y_L) \leq F(Y_L)$*. Then we have the following two equilibria* (τ_1^*, τ_2^*)*:*

(i) $\tau_1^* = \inf\{t \geq 0 : Y_t \geq Y_L\}$ *and* $\tau_2^* = \inf\{t \geq 0 : Y_t \geq Y_*\}$, *and*

(ii) $\tau_1^* = \inf\{t \geq 0 : Y_t \geq Y_*\}$ *and* $\tau_2^* = \inf\{t \geq 0 : Y_t \geq Y_L\}$.

Again, we contrast the timing decision of an information monopoly and a leader in an equilibrium of a war of attrition. This time, as intuitively expected, the leader in a war of attrition will not acquire information earlier than an information monopoly. However, the result is slightly stronger. The leader always concedes at Y_L in any equilibrium. Because $Y_L \geq Y_M$ and the inequality is strict for some cases, therefore the leader would acquire information strictly later than an information monopoly sometimes in the presence of the information externality.

[1] Note that, if such Y_P exists, it must be unique by the same argument of Lemma 2.4.4, but if it does not exist, then $Y_* = \max\{Y_{F,1}, Y_{F,0}\}$.

Chapter 3

Ambiguity Aversion in Sequential Trading: Informational Cascades, Mixed Equilibria, and Informativeness of No-trade

3.1 Introduction

There is plentiful information and traders can acquire information from many sources, but traders may still have *ambiguous* beliefs about the value of an asset. We say a belief is *ambiguous* if there is no single probability distribution to describe the uncertainty. The situation of lacking knowledge of a unique probability distribution is reminiscent of the celebrated Ellsberg (1961) paradox. When agents choose between bets based on draws from an urn with a known distribution of balls of different colors and an urn with an unknown distribution, they are more likely to choose bets with known odds over the bets with unknown odds on the same stakes, i.e. they exhibit *ambiguity aversion*. This observation is inconsistent with the framework of subjective expected utility. Trading an asset is betting on the valuation of the asset. Therefore, ambiguity aversion can affect the decisions of both traders and market makers when they have ambiguous beliefs. This motivates us to study the effects of ambiguity aversion in an order-driven trading mechanism à la Glosten and Milgrom (1985).

Informational cascades (Banerjee, 1992; Bikhchandani et al., 1992) occur when the agent acts by conforming to the public belief and ignoring the private information. This informational externalities of the social learning process provides a potential explanation of widely observed herding behavior. Typically price is necessary to be inflexible so that an information cascade can occur. However, in the Glosten and Milgrom (1985) trading mechanism, the competitive market maker can learn from the traders and flexibly adjust the price aligning with the public belief. Therefore, it is generally difficult to generate informational cascades in sequential trading mechanism, as argued by Avery and Zemsky (1998).

We show that informational cascades can happen in the equilibrium of the sequential trading mechanism if beliefs are ambiguous about the underlying value of an asset. If the agent's belief is sufficiently ambiguous about the value of an asset, then she will neither buy nor sell this asset for a wide range of prices (see Dow and da Costa Werlang 1992). This no-trade phenomena can also occur in our trading environment. When the public beliefs are sufficiently ambiguous, the worst valuation of an agent with a good private signal is lower than the ask price and the best valuation of an agent with a bad private signal is higher than the bid price. Therefore, no matter what private signal an informed trader gets, she will be reluctant to trade and, consequently, an information cascade on no-trade can occur.

Moreover, ambiguity aversion can reinforce the problem of adverse selection and contribute to the ask-bid spread. The market maker asks a higher price to sell the asset and bids a lower price to buy the asset to compensate for their aversion to ambiguity. Therefore, when public belief in the market is more ambiguous, the ask-bid spread becomes wider.

To formalize these ideas, we study a trading environment that is similar to the special case of Glosten and Milgrom (1985) but with ambiguity averse agents. There are three types of agent, noise traders, informed traders, and the market maker. Trading is sequential and it is conducted in an exogenous order. The noise traders trade inelastically and randomly. The informed traders receive some private signals about the value of the asset and make their trading decisions rationally. The market maker is risk neutral and sets ask and bid prices to elicit informed traders to trade according to their private signals. The multiple prior expected utility (Gilboa and Schmeidler, 1989) is introduced to capture the ambiguity aversion of both the market maker and informed traders.

The existence of equilibrium is established in a general model, and we also show that informational cascades can exist in equilibrium under certain conditions. It is attributed to ambiguity aversion and is restricted to the case that all the informed agents choose not to trade because of their aversion to uncertainty. One consequence of an information cascade is that it overturns the convergence result of Glosten and Milgrom (1985): Once an informational cascade starts, there is no further information revealed in the equilibrium path about the value of an asset, and hence, prices stay the same and ask-bid spreads remain forever.

To fully characterize the equilibrium and illustrate effects of ambiguity aversion in sequential trading, the general model is simplified to a binary model, where the value of the asset and the signals are binary. We compare the equilibrium of the binary models under ambiguity with the model where there is no ambiguity.

Apart from the possibility of informational cascades in equilibrium, a surprising result is that there could be no equilibrium in pure strategies when there is a certain level of ambiguity. This is in contrast to the existence of equilibrium in pure trading strategies when there is no ambiguity in the binary model. If there is too much ambiguity, then there will be informational cascades as discussed. The amount of ambiguity needs to be low enough so that informed traders will buy or sell with probability one upon receiving a good or bad signal, respectively. However, if the amount of ambiguity is in between, there can be no equilibrium in pure strategies and no pricing rule is consistent with market maker's conditions.

The introduction of mixed trading strategy is necessary to guarantee the existence of equilibrium in the trading environment under ambiguity aversion. When mixed trading strategies are played in equilibrium, the noise-to-signal ratio of trading activities is increased. Therefore convergence of beliefs and prices can take much longer.

The comparative statics of the effect of ambiguity on the ask-bid spread is also established in the binary model. The ask-bid price widens when there is more ambiguity in the market. As previously mentioned, to compensate for market maker's aversion to ambiguity, the ask price will be set higher and the bid price lower when belief becomes more ambiguous.

A more interesting result emerges when the private signals are ambiguous. If we assume that initially there is no informational cascade, then the market maker is supposed to set ask and bid prices to elicit informed traders to reveal their private signal imperfectly. Whenever the private signal is revealed, its ambiguity is added to the public beliefs. Therefore, the public belief becomes more ambiguous. Consequently, the market condition would be arbitrarily close to an informational cascade.

Lastly, we consider a binary model with asymmetric binary signals in the sense that a good signal is more informative than a bad signal under ambiguity. The informed traders play mixed trading strategies in the equilibrium, therefore informed traders can also choose not to trade even when there is no informational cascade. When the bad signal is less informative than the good signal, an informed trader with a bad signal is more likely to choose not to trade than an informed trader with a good signal. Thus, no news is bad news: when no-trade occurs in the market, the valuation of the asset will decrease.

Briefly, from an empirical point of view, the results of this paper can be linked to some phenomena of trading activities regarding some illiquid assets. For example, there are some stocks traded with low volume daily and typically they are also barely analyzed and covered in the media, except for their earnings announcements. Or, for some other stocks, there are some periods in which trading volume is low and information about the stock is very limited. Usually, we will see that bid-ask spreads are wide for these illiquid stocks and that they show no sign of convergence. In such a scenario, this paper can provide an explanation on the basis of uncertainty aversion.

This paper contributes to relate ambiguity aversion and information cascade in financial market. Ambiguity aversion reduces traders' incentive to trade. Dow and da Costa Werlang (1992) first show a no-trade under ambiguity. The no-trade is the key to generate information cascade in this chapter. In the literature, without the ambiguous information, Avery and Zemsky (1998) conclude the impossibility of information cascade in sequential trading mechanism. Instead, they introduce multidimensional uncertainty and show that there can be a herding behavior. Park and Sabourian (2011) also show that such herding behavior can occur if there are at least three possible values of an asset. Different from information cascade, the definition of herding in their paper states that traders follow the crowds in the sense that their trading decisions are altered upon the observation of public history, and yet there can still be social learning and prices converge to the true value. But our definition of information cascade follows; social learning stops and prices never converge to the truth when an information cascade occurs. Some papers also obtain the information cascade in sequential trading mechanism under other assumptions such as transaction costs (Lee, 1998), exogenous gains and loss from trade (Cipriani and Guarino, 2008), risk aversion of traders and discrete actions (Decamps and Lovo, 2006), and so forth.

The paper is organized as follow. Section 3.2 describes the general specification of the model, establishes the existence of equilibrium, and show that information cascades exhibit in some equilibrium. Section 3.3 analyzes a simple example, namely the binary models, and we contrast the results with and without ambiguity and fully characterize the equilibrium with mixed trading strategies. Then, Section 3.4 extends the binary models with mixed trading strategy to show two additional results: (i) The market tends to be arbitrarily close to an informational cascade if private signals are ambiguous; and (ii) no-trade sometimes can be informative. Finally, Section 3.5 concludes.

3.2 The General Model

We start by describing the *trading game* of interest, which is similar to that of Avery and Zemsky (1998) and a special case of Glosten and Milgrom (1985). Time is discrete and denoted by $t = 1, 2, \ldots, T$ with $T \leq \infty$. Let $V \subset \mathbb{R}_+$ be a finite (but non-singleton) set of possible values of an asset, and the true value of the asset $v \in V$ is not observable until the end of period T. Ambiguity for both informed traders and market maker is introduced in the model à la Gilboa and Schmeidler (1989), that is, there is not a single probability measure, or belief, to describe the uncertainty, but a closed and convex set $\Pi_1 \subseteq \Delta(V)$ that collects all the beliefs that agents think possible. The following is a simple example:

Example 3.2.1. Let's consider $V = \{0, 1\}$ so that the $\Delta(V)$ is equivalent to $[0, 1]$. Let $\Pi_1 = [\underline{\pi}_1, \bar{\pi}_1] \subseteq \Delta(V)$ with $\underline{\pi}_1 < \bar{\pi}_1$, where $\pi_1 = \Pr(v = 1)$ for all $\pi_1 \in \Pi_1$.

There are three types of agents in the model, a market maker, informed traders, and noise traders. The trading arrangement is as follow: think of some period t, and let history h_t and the set $\Pi_t(h_t)$ of common beliefs, both of which will be specified later, be given. First, the market maker posts ask price a_t and bid price b_t. Then the market marker meets either an informed trader with probability $\eta \in (0, 1)$ or a noise trader with probability $1 - \eta$. Each trader can either buy or sell one unit of asset, or choose not to trade, recorded as $z_t = 1, -1,$ or 0, respectively. To clarify, "buy" and "sell" is defined from the perspective of the trader, not the market maker, e.g. $z_t = 1$ means that a trader buys one unit of asset from the market maker, or, equivalently, the market maker sells one unit of asset to the trader at the ask price. The noise traders are assumed to trade for some exogenous reasons and their demand for the asset is inelastic. Simply they buy, or to sell one unit of the asset, or not to trade with equal probabilities, i.e. 1/3.

In contrast, the informed traders act rationally and choose the probability of trading $\sigma_t = (\zeta_t, \iota_t)$ in period t, where ζ_t and ι_t are the probabilities of buying and selling, respectively. Informed traders are playing mixed strategies instead of pure strategy, i.e. simply choosing to buy, or sell, or not to trade. Apart from the generality, this assumption is essential for the existence of equilibrium.

The informed trader who meets the market maker at period t receives an independent private signal $s_t \in S \subseteq \mathbb{R}$. Assume that S is a non-singleton and finite set. Let $p(\cdot|\cdot) : S \times V \to [0, 1]$ be the conditional probability of signal, i.e. $p(s|v) = \Pr(s_t = s|v)$ is the probability of signal $s_t = s$ conditional on that the true value is v. Signals can also be ambiguous. The informed agent considers a set of conditional probabilities $\mathcal{P}$, which is closed and convex. That the set $\mathcal{P}$ is constant over time can be justified by the assumption that the ambiguous signals are *conditionally independent* in the sense that they are *epistemically independent* (Couso et al., 1999) conditional on value $v \in V$ of the asset. Loosely speaking, knowing the past signals will not change the perceived set of conditional probabilities of the future signals. Moreover, the joint distribution of value v and signal s at period t is $\mathcal{Q}_t = \{p \times \pi : \pi \in \Pi_t, p \in \mathcal{P}\}$.

Example 3.2.2. Let's consider that $V = \{0,1\}$ and $S = \{0,1\}$ and let $\mathcal{P} = [\underline{q}, \bar{q}]$ where $1/2 < \underline{q} < \bar{q} < 1$. Two extreme conditional distributions are:

Table 3.2.1: Conditional probabilities

	$\Pr(s_t = 1\|v)$	$\Pr(s_t = 0\|v)$
$v = 1$	$\underline{q}$	$1 - \underline{q}$
$v = 0$	$1 - \underline{q}$	$\underline{q}$

	$\Pr(s_t = 1\|v)$	$\Pr(s_t = 0\|v)$
$v = 1$	$\bar{q}$	$1 - \bar{q}$
$v = 0$	$1 - \bar{q}$	$\bar{q}$

Public History, Beliefs, and Equilibrium

The *history* h_t observable to all the agents has length $t-1$ and contains the sequence of prices $\{a_\tau, b_\tau\}_{\tau=1}^{t-1}$ and trading orders $\{z_\tau\}_{\tau=1}^{t-1}$, i.e. $h_t = \{a_\tau, b_\tau, z_\tau\}_{\tau=1}^{t-1}$. H_t denote the set of all possible histories with length $t-1$, and as a convention, history with length 0 is an empty set, i.e. $h_1 = \emptyset$.

Without access to private signals, the market marker's *pricing strategy* $x_t = (a_t, b_t) : H_t \to \mathbb{R}_+^2$ are measurable w.r.t. public histories, where $a_t(h_t)$ and $b_t(h_t)$ are the ask and bid prices, respectively, posted at period t; and denote $x = (x_t)_{t=1}^T$ and X to be the set of all the possible pricing strategies. The informed trader at period t has the *trading strategy* $\sigma_t = (\zeta_t, \iota_t) : H_t \times \mathbb{R}_+^2 \times S \to \Delta$ that are measurable w.r.t. history, ask and bid prices, and the private signal received, where $\zeta_t(h_t, a_t, b_t, s_t)$ and $\iota_t(h_t, a_t, b_t, s_t)$ are the probabilities of buying and selling, respectively, at period t following history h_t and given the ask a_t, bid b_t, and the private signal s_t.[1] Denote Σ_t to be the set of all the possible trading strategies at period t for informed traders.

The informed trader's problem is to maximize the *min-expected utility* as follow:

$$\max_{\sigma_t \in \Sigma_t} \min_{p \in \mathcal{P}, \pi \in \Pi_t(h_t)} E^{p \times \pi} \left[\zeta_t u(v - a_t) + \iota_t u(b_t - v) | h_t, a_t, b_t, s_t \right]. \tag{3.2.1}$$

[1]Here the Δ denotes the simplex in $\mathbb{R}^3$. Since $(\zeta, \iota) \in \mathbb{R}^2$, so more precisely $\Delta = \{(\zeta, \iota) \in [0,1] \times [0,1] : \zeta + \iota \leq 1\}$ is homeomorphic to a simplex in $\mathbb{R}^3$.

Let us assume that market maker's objective is to set prices so as to maximize the *min-expected profit* of each trade, either a "buy" or a "sell", at each period. To justify this assumptio, think of that the market maker is myopic (i.e. not considering its asset position but only caring about the current trade or they are short-lived for only one period with zero intial exposure to the uncertain asset). Moreover, because of the competition between market makers assumed in the background, its min-expected profit has to be zero, hence its equilibrium conditions are just reduced to a set of *zero min-expected profit* conditions, i.e.,

$$\min_{p\in\mathcal{P},\pi\in\Pi_t(h_t)} E^{p\times\pi}\left[a_t - v|h_t, a_t, b_t, z_t = 1\right] = 0, \tag{3.2.2}$$

$$\min_{p\in\mathcal{P},\pi\in\Pi_t(h_t)} E^{p\times\pi}\left[v - b_t|h_t, a_t, b_t, z_t = 0\right] = 0. \tag{3.2.3}$$

These set of zero min-expected profit conditions parallel to zero profit conditions of the market maker in other literature on order-driven tradings: if both $\mathcal{P}$ and $\Pi(h_t)$ are singleton sets, then it reduces to the set of usual zero expected profit conditions.

The *belief system* $\Pi = (\Pi_t)_{t=1}^T$ is a collection of correspondences $\Pi_t : H_t \rightrightarrows \Delta(V)$ for all $t > 1$ with set Π_1 given, where $\Pi_t(h_t)$ denotes the set of *public* (or *common*) *beliefs* at the beginning of period t following history h_t. Given some history h_t, $t > 1$, a belief $\pi_t \in \Delta(V)$ at history h_t is *consistent* w.r.t. $\pi_1 \in \Pi_1$, $p \in \mathcal{P}$, and the strategy profile (x, σ), if it is updated by Bayes rule wherever it is possible. We do this inductively, specifically, for any $h_2 = (a_1, b_1, z_1)$, π_2 is consistent if it satisfies

$$\pi_2(v) = \frac{\pi_1(v)\Pr(z_1|v)}{\sum_{v'}\pi_1(v')\Pr(z_1|v')}.$$

For any $t > 1$ and any history $h_{t+1} = (h_t, a_t, b_t, z_t)$, π_{t+1} is consistent if, π_τ is consistent for all $\tau \leq t$ and

$$\pi_{t+1}(v) = \frac{\pi_t(v)\Pr(z_t|h_t, v)}{\sum_{v'}\pi_t(v')\Pr(z_t|h_t, v')},$$

where

$$\Pr\left(z_t|h_t,v\right)=\begin{cases}\frac{1-\eta}{3}+\eta\sum_{s_t\in S}\zeta\left(h_t^x,s_t\right)p(s_t|v) & \text{if } z_t=1\\ \frac{1-\eta}{3}+\eta\sum_{s_t\in S}\iota\left(h_t^x,s_t\right)p(s_t|v) & \text{if } z_t=-1\end{cases}.$$

A remark on notations: if the pricing strategy x is specified, then we denote the *augmented history* $h_t^x=(h_t,a_t(h_t),b_t(h_t))$, e.g. $\zeta\left(h_t^x,s_t\right)=\zeta\left(h_t,a_t(h_t),b_t(h_t),s_t\right)$.

The belief system Π is consistent if for every h_t and $\pi_t\in\Pi_t(h_t)$, π_t is consistent w.r.t. some initial belief, conditional probabilities of signals, and strategies, which is formally defined as follow.

Definition 3.2.1. The belief system Π is *consistent* w.r.t. Π_1, $\mathcal{P}$, and strategy profile (x,σ), if, for any $t>1$ and history h_t, for any $\pi_t\in\Pi_t(h_t)$, there exists some $\pi_1\in\Pi_1$ and $p\in\mathcal{P}$ such that π_t is consistent w.r.t. π_1, p, and (x,σ).

We define an equilibrium in the spirit of perfect Bayesian equilibrium, constituted by a pair of strategies and a belief system, where strategies are best responses given the beliefs and beliefs are consistent given the strategies.

Definition 3.2.2. An *equilibrium* of this game consists of a strategy profile (x,σ) and the system of beliefs Π such that:

(i) given the belief system Π and trading strategy σ, the pricing strategy x satisfies the zero min-expected profit conditions (3.2.2) and (3.2.3);

(ii) given the belief system Π and pricing strategy x, the trading strategy σ maximizes the min-expected utility given any signal $s_t\in S$, i.e.

$$\sigma_t(h_t^x,s_t)\in\arg\max_{\sigma\in\Sigma}\min_{p\in\mathcal{P},\pi\in\Pi_t(h_t)}E^{p\times\pi}\left[\zeta_t u\left(v-a_t(h_t)\right)+\iota_t u\left(b_t(h_t)-v\right)|h_t^x,s_t\right];$$

(iii) the belief system Π is consistent w.r.t. Π_1, $\mathcal{P}$, and (x,σ).

Remark 3.2.1. The conditional probabilities p satisfies MLRP implies that $\Pr\left(\cdot|s'\right)$ first-order stochastically dominates (FOSD) $\Pr\left(\cdot|s\right)$ if $s'>s$.

Note that FOSD relation is a consequence of MLRP. Example 3.2.2 satisfies assumption 2. And either FOSD or MLRP captures that the signals are informative, i.e. larger signals are better in the sense that they imply higher probability of high value of the asset.

We characterize the pricing rules of the market maker, which has been extensively studied in the case without ambiguity, that is, the ask and bid prices are the expected value of the asset conditional on a "buy" or a "sell" in this period, respectively. However, under ambiguity this result is slightly modified in a way that the determination of ask and bid prices relies on different beliefs in the set of consistent beliefs. The intuition is that, because of fear of uncertainty, when there is a "buy", the market maker wants to sell the asset at the highest conditional expected value, and, when there is a "sell", the market maker, conversely, wants to buy the asset at the lowest conditional expected value.

Lemma 3.2.1 (Pricing Strategies). *Equilibrium pricing rule x_t satisfies the following*

$$a_t(h_t) = \max_{p \in \mathcal{P}, \pi \in \Pi_t(h_t)} E^{p \times \pi} \left[v | h_t^x, z_t = 1 \right], \tag{3.2.4}$$

$$b_t(h_t) = \min_{p \in \mathcal{P}, \pi \in \Pi_t(h_t)} E^{p \times \pi} \left[v | h_t^x, z_t = -1 \right], \tag{3.2.5}$$

for any history h_t in equilibrium.

We omit the proof because it follows straightforwardly from equations (3.2.2) and (3.2.3).

Theorem 3.2.1. *There exist an equilibrium (x, σ, Π) for the trading game.*

Proof. In the Appendix to Chapter 3. □

In Section 3.3, models with binary states and signals are presented. We contrast the binary models with and without ambiguity: If there is no ambiguity in the market, then there exists a unique equilibrium where informed trader employ pure strategies, however, if there is ambiguity in the market, then there cannot exist an equilibrium where informed traders only employ pure strategy. It is essential to guarantee the existence of equilibrium to admit mixed trading strategies by informed traders.

Before further analysis of the properties of an equilibrium, let us impose some assumptions throughout the paper.

Assumption 1. *u is concave and strictly increasing, and $u(0) = 0$.*

Assumption 2. *Any conditional probabilities $p \in \mathcal{P}$ satisfies the monotone likelihood ratio property (MLRP), i.e.*

$$\frac{p(s'|v')}{p(s'|v)} \geq \frac{p(s|v')}{p(s|v)},$$

whenever $s' \geq s$ and $v' \geq v$, with strict inequality if $s' > s$, or $v' > v$, or both.

3.2.1 Informational Cascade in Equilibrium

The concept of informational cascade, first termed by Bikhchandani et al. (1992), says that the informed agent forgoes her private information in the sense that her optimal decision mimics her predecessor's decision and is independent of her private signal. Formally, it is defined as follow.

Definition 3.2.3. Given the strategy profile (x, σ), there is an *informational cascade* after history h_t if the trading strategy of an informed trader at h_t is independent of the private signal, i.e.

$$\sigma_t\left(h_t^x, s_t\right) = \sigma_t\left(h_t^x, s_t'\right)$$

for all $s_t, s_t' \in S$.

Avery and Zemsky (1998) show that it is impossible to have informational cascades in equilibrium of a trading game *without* ambiguity. However, it is overturned in a trading game *with* ambiguity.

Given an equilibrium (x, σ, Π), it induce a probability distribution on H_t, for all $t \leq T$, and is denoted by $\Pr(\cdot; x, \sigma, \Pi)$. Let the support $H_t(x, \sigma, \Pi) = \{h_t \in H_t | \Pr(h_t; x, \sigma, \Pi) > 0\}$ denote the history before period t that is reachable in the equilibrium (x, σ, Π), or the *equilibrium path*, and $h_t \in H_t(x, \sigma, \Pi)$ is a *history in equilibrium*. Our analysis, especially analysis of informational cascades, will focus on the equilibrium path of the game.

First, there cannot be any no informational cascade where all informed traders buy or sell with strictly positive probability.

Proposition 3.2.1. *For any equilibrium (x, σ, Π), there does not exist any $h_t \in H_t(x, \sigma, \Pi)$ such that:*

1. $\zeta_t(h_t^x, s_t) = \zeta$ *for all* $s_t \in S$*; or*

2. $\iota_t(h_t^x, s_t) = \iota$ *for all* $s_t \in S$

for any $\zeta, \iota > 0$.

Proof. In the Appendix to Chapter 3. □

However, there is still possibility of informational cascade on not trading, i.e. $\zeta_t(h_t^x, s_t) = \iota_t(h_t^x, s_t) = 0$. Indeed, there is a sufficient and necessary condition regarding the set of public beliefs for an informational cascade on no trading.

Proposition 3.2.2. *Given some equilibrium, there is an informational cascade at history h_t if and only if*

$$\min_{p \in \mathcal{P}, \pi \in \Pi_t(h_t)} E^{p \times \pi} \left[u \left(v - \max_{\pi \in \Pi(h_t)} E^{\pi} [v | h_t] \right) | h_t, \bar{s} \right] \leq 0, \text{ and}$$
$$\min_{p \in \mathcal{P}, \pi \in \Pi_t(h_t)} E^{p \times \pi} \left[u \left(\min_{\pi \in \Pi(h_t)} E^{\pi} [v | h_t] - v \right) | h_t, \underline{s} \right] \leq 0,$$

where $\underline{s} = \min S$, $\bar{s} = \max S$. Moreover, in this informational cascade,

$$\zeta_t(h_t^x, s_t) = \iota_t(h_t^x, s_t) = 0$$

for all $s_t \in S$.

Proof. In the Appendix to Chapter 3. □

The necessary and sufficient condition in the Proposition 3.2.2 is somewhat difficult to interpret. Therefore, we give Theorem 3.2.2, which is simply an implication of Jensen's inequality. However, it provides a easy-to-verify and easy-to-interpret sufficient condition that guarantees the informational cascades on not trading.

Theorem 3.2.2. *Given some equilibrium (x, σ, Π), there is an informational cascade at some history h_t, in which $\zeta_t(h_t^x, s_t) = \iota_t(h_t^x, s_t) = 0$, $\forall s_t \in S$, if*

$$\min_{p \in \mathcal{P}, \pi \in \Pi_t(h_t)} E^{p \times \pi}\left[v | h_t, \bar{s}\right] \leq \max_{\pi \in \Pi(h_t)} E^{\pi}\left[v | h_t\right], \text{ and} \tag{3.2.6}$$

$$\min_{p \in \mathcal{P}, \pi \in \Pi_t(h_t)} E^{p \times \pi}\left[v | h_{,t}, \underline{s}\right] \geq \min_{\pi \in \Pi(h_t)} E^{\pi}\left[v | h_t\right]. \tag{3.2.7}$$

Proof. In the Appendix to Chapter 3. □

For example, look at the inequality (3.2.6): If the difference between $\max_{\pi \in \Pi(h_t)} E^{\pi}\left[v | h_t\right]$ and $\min_{\pi \in \Pi(h_t)} E^{\pi}\left[v | h_t\right]$ is large, which means that the ambiguity in the market is large, and if the difference between $\min_{p \in \mathcal{P}, \pi \in \Pi_t(h_t)} E^{p \times \pi}\left[v | h_t, \bar{s}\right]$ and $\min_{\pi \in \Pi(h_t)} E^{\pi}\left[v | h_t\right]$ is small, which means that the informativeness of the signal is relatively small, then typically the inequality will hold. In the binary model studied in Section 3.3, it is more clear how this type of informational cascade arises when the ambiguity is greater than the informativeness of the signal.

3.2.2 The Characterization of Equilibrium

First, the next proposition just extends the result of endogenous positive ask-bid spread into this ambiguous trading game.

Proposition 3.2.3. *In an equilibrium* (x, σ, Π)*, for every* $h_t \in H_t(x, \sigma, \Pi)$*,*

$$a_t(h_t) > b_t(h_t).$$

Proof. In the Appendix to Chapter 3. $\square$

Here, we will briefly comment on the asymptotics of the equilibrium. It may be interesting to know whether it can be the case that the market tends to an informational cascade when the private signals are ambiguous when it starts out with no informational cascade. This is possible because the ambiguity of the private signals is adding to common beliefs. However, proving this in the general model is a difficult task. Instead, we will show this in the binary model with ambiguous signals in Section 3.4. Nevertheless, in the case that there is an informational cascade, the dynamics of the prices and beliefs will simply be constant.

Corollary 3.2.1. *Given an equilibrium* (x, σ, Π)*, if there is an informational cascade after history* h_t *in equilibrium, then there are informational cascades for all histories* h_τ*,* $\tau \geq t$*, that follow history* h_t *in equilibrium, and the equilibrium ask bid spreads,* $a(h_\tau) - b_\tau(h_\tau)$*, remains constant, for all* $\tau \geq t$*, and strictly positive.*

Proof. In the Appendix to Chapter 3. $\square$

To characterize an equilibrium, there can be either informational cascade or no cascade in each round of trading. When there is no cascade, the trading decisions in equilibrium are *monotonic*, that is, it can be describe as such: there are two crucial signals, $s^* > \underline{s}$ and $s_* < \bar{s}$, with $s_* < s^*$, and the active informed traders will buy the asset if her private signal is at least s^* and sell the asset if her private signal is no more than s_*, where the determination of s^* and s_* depends on the set of common beliefs at that period.

Theorem 3.2.3. *In any equilibrium* (x, σ, Π)*, given any* $h_t \in H_t(x, \sigma, \Pi)$*, note that one of the following cases must hold:*

(i) $\zeta_t(h_t^x, s_t) = 0$ *and* $\iota_t(h_t^x, s_t) = 0$ *for all* $s_t \in S$ *(informational cascade), or*

(ii) There exists some $\underline{s} < s_t^* \leq \bar{s}$ *such that*

$$\zeta_t(h_t^x, s) \begin{cases} = 0 & \forall s < s_t^* \\ > 0 & s = s_t^* \\ = 1 & \forall s > s_t^* \end{cases}$$

or

(iii) There exists some $\underline{s} \leq s_{t*} < \bar{s}$ *such that*

$$\iota_t(h_t^x, s) \begin{cases} = 1 & \forall s < s_{t*} \\ > 0 & s = s_{t*} \\ = 0 & \forall s > s_{t*} \end{cases} ;$$

and (ii) and (iii) can hold simultaneously with $s_{t*} < s_t^*$*.*

Proof. In the Appendix to Chapter 3. □

3.3 The Binary Model

In this section, we study some simplified version of the model and call them binary models because the set V of the possible value of the asset and set S of possible signals are assumed to take only two values, i.e. $V = S = \{0, 1\}$. Moreover, T is set to be infinity, and, to simplify the analysis, the utility functions of the informed agents are assumed to be linear, i.e. $u(x) = x$; hence, they are risk neutral. First, we present the model without ambiguity, where there exists a unique equilibrium with pure trading strategies and informational cascade is impossible in equilibrium. Then, ambiguity is introduced in the minimal way. Under ambiguity, there can be informational cascade and equilibrium may not exist if traders only employ pure trading strategies. Hence, mixed trading strategies are essential to guarantee the existence of equilibrium.

3.3.1 The Binary Model without Ambiguity

Without ambiguity, the information structure in the basic model is simple. Initially, there is only one initial common belief $\pi_1 = \Pr(v = 1)$, and there is only one conditional distribution of a symmetric binary private signal,

$$p(1|1) = p(0|0) = q,$$

with $q > 1/2$, so that the signal structure satisfies MLRP, or intuitively, 1 is a good signal since it implies higher valuation of the asset and, conversely, 0 is a bad signal. Moreover, let's now assume that the traders can only choose pure strategy, i.e. the investment decision $\hat{z}_t \in Z = \{-1, 0, 1\}$. This is equivalent to restrict the probabilistic trading strategies chosen within the degenerate distributions of actions, i.e. $\sigma(h_t, a_t, b_t, s_t) \in \{(0,1), (0,0), (1,0)\}$.

The market maker's conditions are reduced to the conventional zero expected profit conditions, and the equilibrium is a perfect Bayesian equilibrium if we assume that the zero expected profit conditions are derived from profit maximization problem under competition. Therefore, the *(perfect Bayesian) equilibrium* is then, a pair of strategies and beliefs, such that, given the belief system $\pi = (\pi_t)_{t=1}^T$, the price strategies, sequence of functions $\{a_t, b_t : H_t \to \mathbb{R}_+\}$, satisfies the zero expected profit condition and trading strategies, sequence of functions $\left\{\hat{z}_t : H_t \times \mathbb{R}_+^2 \times \{0,1\} \to Z\right\}$, maximizes the informed agent's expected payoff for those who are active at period t, and beliefs are consistent in the sense that the conditional beliefs $\{\pi_t : H_t \to [0,1]\}$ satisfy the Bayes rule wherever it is possible.

Given history h_t and the public belief $\pi_t(h_t)$ at the beginning of period t, or simply denoted as π_t; and $\pi_t^{(\cdot)} : S \to [0, 1]$ is the *updated belief* of the informed traders where

$$\pi_t^{(s_t)} = \frac{\pi_t p(s_t|1)}{\pi_t p(s_t|1) + (1 - \pi_t) p(s_t|0)}.$$

Hence, the interpretation of $\pi_t^{(s_t)}$ is the posterior belief given prior π_t and signal s_t.

Trading games of this specification have been studied extensively in the previous literature. There is detailed analysis in Avery and Zemsky (1998), and the binary model here follows closely the exposition in Chamley (2004). First, there exists some equilibrium for this trading game. And, in the equilibrium, the market maker typically post different ask and bid prices, which generates endogenous bid-ask spread. Moreover, in the equilibrium, informed traders separate themselves by trading orders: traders with good signal (i.e. $s_t = 1$) will buy the asset and traders with bad signal (i.e. $s_t = 0$) will sell the asset. These features are then summarized in the following proposition.

Proposition 3.3.1. *The equilibrium exists. Moreover, in equilibrium,*

(i) $\pi_t^{(0)} < b_t(h_t) < \pi_t < a_t(h_t) < \pi_t^{(1)}$*; and*

(ii) $\hat{z}_t(h_t^x, 1) = 1$ *and* $\hat{z}_t(h_t^x, 0) = -1$*,*

for every history h_t*.*

Proof. In the Appendix to Chapter 3. □

First, we can show that it is only possible for the informed trader to buy with a good signal and sell with a bad signal, as (ii) in the Proposition 3.3.1. Anticipating this, consequently, the market maker's conditions would be

$$\frac{1-\eta}{3}\left(a_t(h_t) - \pi_t\right) + \eta\left[\pi_t q\left(a_t(h_t) - 1\right) + (1 - \pi_t)(1 - q)a_t(h_t)\right] = 0$$
$$\frac{1-\eta}{3}\left(\pi_t - b_t(h_t)\right) + \eta\left[\pi_t(1 - q)\left(1 - b_t(h_t)\right) - (1 - \pi_t)q b_t(h_t)\right] = 0,$$

which yields the unique solution that

$$a_t(h_t) = \frac{\frac{1-\eta}{3\eta}\pi_t + \pi_t q}{\frac{1-\eta}{3\eta} + \pi_t q + (1-\pi_t)(1-q)} \in (\pi_t, \pi_t^{(1)}), \tag{3.3.1}$$

$$b_t(h_t) = \frac{\frac{1-\eta}{3\eta}\pi_t + \pi_t(1-q)}{\frac{1-\eta}{3\eta} + \pi_t(1-q) + (1-\pi_t)q} \in (\pi_t^{(0)}, \pi_t), \tag{3.3.2}$$

and hence (i) in the proposition follows.

Then, as a corollary, there cannot be informational cascade in the equilibrium path since the informed traders will trade differently upon receiving different signals. In general, it is true that if there is no ambiguity then there is no informational cascade.

Else, other properties of the equilibrium, such as the dynamics of the public belief and prices, are the same as that in the previous literature. Especially, the public belief π_t is a martingale in equilibrium. Without informational cascades, private information is revealed, in the long run, as Glosten and Milgrom (Proposition 4, 1985), so that π_t will converge to the true value almost surely. Moreover, there is an endogenous bid-ask spread, however, both a_t and b_t will also converge to the true value, and bid-ask spread will converge to 0, almost surely.

Finally, Figure 3.3.1 illustrates an example of the dynamics for prices and the public beliefs, where we set $v = 1$, $\eta = 0.5$, $\pi_1 = 0.5$ and $q = 0.6$. It can be seen that the convergence of prices and beliefs typically happens after roughly 250 rounds of trading.

3.3.2 The Binary Model with Ambiguous Prior

In this section, ambiguity is introduced in the binary model. If the amount of ambiguity is low, as the case where there is no ambiguity, the equilibrium will be just like the one in the previous section. However, otherwise we will see two quite different results compared with the situation without ambiguity: If the amount of ambiguity is sufficiently large, then the equilibrium exhibits informational cascades.

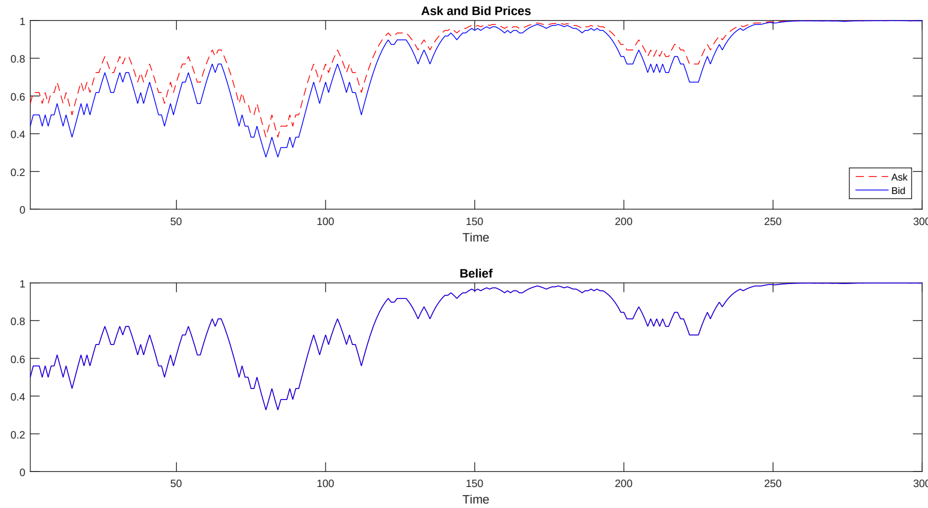

Figure 3.3.1: A sample path of prices and beliefs without ambiguity

When the amount of ambiguity is somewhat in between, we will see that surprisingly there cannot be any equilibrium with pure trading strategies.

First, the public beliefs are ambiguous. Initially, instead of one common belief, there is a set of common beliefs as Example 3.2.1, $\Pi_1 = [\underline{\pi}_1, \bar{\pi}_1]$ with $\underline{\pi}_1 < \bar{\pi}_1$. For notation matters: given some belief system Π, for any $\Pi_t(h_t)$, let $\underline{\pi}_t = \min \Pi_t(h_t)$ and $\bar{\pi}_t = \max \Pi_t(h_t)$. $\pi_t^{(s)}$ is defined as that in the previous section. Moreover,

$$\Pi_t^{(s)} := \left\{ \pi_t^{(s)} | \, \pi_t \in \Pi_t(h_t) \right\}.$$

The notion of equilibrium (with pure strategies) here is consisting of pricing strategies $x_t = (a_t, b_t) : H_t \to \mathbb{R}_+^2$, trading strategies $\hat{z}_t : H_t \to \{-1, 0, 1\}$, and the belief system Π as defined in the general model, such that pricing strategies solves the zero min-expected profit condition as that in the general model, and trading strategies are optimal given prices and private signal, and the belief system is consistent.

Then, the public beliefs $\Pi_t(h_t)$ in each period will be typically a set, and the trading strategies will be slightly different from that in the previous section. Notice that the min-expected payoff of buying an asset is

$$\min_{\pi \in \Pi_t^{(s)}} [\pi - a_t(h_t)]$$

and the min-expected payoff of selling an asset is

$$\min_{\pi \in \Pi_t^{(s)}} [b_t(h_t) - \pi] .$$

Especially, given the private signal s, if least optimistic valuation of the trader is less than the ask price, i.e.

$$\min \Pi_t^{(s)} \leq a_t(h_t),$$

then the informed trader with signal s will not buy the asset, and if the most optimistic valuation is greater than the bid price, i.e.

$$\max \Pi_t^{(s)} \geq b_t(h_t),$$

then the informed trader with signal s will not sell the asset. So when both hold, there will not be any trading orders from that informed trader. But if either one of the above inequalities is reversed, then the trader will trade in the favorable direction. This proves the following lemma that summarizes the pure trading strategies under ambiguity.

Lemma 3.3.1 (Pure Trading Strategies)**.** *For the informed trader, who is active immediately after history h_t and receiving signal s_t, buying the asset is optimal if and only if*

$$\min \Pi_t^{(s)} \geq a_t(h_t),$$

and selling the asset is optimal if and only if

$$\max \Pi_t^{(s)} \leq b_t(h_t),$$

and not to trade is optimal if and only if

$$\min \Pi_t^{(s)} \leq a_t(h_t) \text{ and } \max \Pi_t^{(s)} \geq b_t(h_t).$$

Before further analysis of the equilibrium, the following lemma is useful to show that, in the equilibrium, there can be only two situations: one is an informational cascade with no trade, and the other is just like the case without ambiguity, i.e. traders separating themselves in trading upon receiving different signals.

Lemma 3.3.2. *For every history h_t in equilibrium (with pure strategies), trading orders $\{\hat{z}_t\}$ satisfies one of the following conditions:*

(i) $\hat{z}_t(h_t^x, 1) = \hat{z}_t(h_t^x, 0) = 0$*; or*

(ii) $\hat{z}_t(h_t^x, 1) = 1$ *and* $\hat{z}_t(h_t^x, 0) = -1$.

Proof. In the Appendix to Chapter 3. □

The following proposition states that, when the public beliefs are ambiguous enough, then there has to be an informational cascade. The intuition is as follow. Suppose that we have informational cascade, specifically no informed traders trading given the prices with any private signal, then because of zero min-expected profit condition, the ask and bid prices will be set to be $\bar{\pi}_t$ and $\underline{\pi}_t$, respectively. If the public beliefs are very ambiguous in the sense that the difference between $\bar{\pi}_t$ and $\underline{\pi}_t$ is large, then it is possible to have $\underline{\pi}_t^{(1)} \leq \bar{\pi}_t = a_t(h_t)$ and $\bar{\pi}_t^{(0)} \geq \underline{\pi}_t = b_t(h_t)$, i.e. the worst valuation with a good signal is still less than the ask price and the best valuation with a bad signal is still greater than the bid price. Hence, no matter what the signal is, it is truly optimal for all informed traders not to trade, that is, an informational cascade hits the market. Consequently, since no information can be revealed in a cascade, the public beliefs are not changing any more. So, in next period, the same situation repeats itself; and informational cascades persist forever.

Proposition 3.3.2. *In an equilibrium,* $\Pi_\tau(h_\tau) = \Pi_t(h_t)$ *and*

$$\hat{z}_\tau(h^x_\tau, s_\tau) = 0$$

for all $\tau \geq t$ *and* $s_\tau \in \{0,1\}$ *if and only if the set of beliefs* $\Pi_t(h_t)$ *satisfies the following:*

$$\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} \geq \frac{q}{1-q}. \tag{3.3.3}$$

Proof. Let the equilibrium be given.

(Only if part.) In period t, let

$$\hat{z}_t(h^x_t, s_t) = 0,\ \forall s_t \in \{0,1\}.$$

Then the market maker's condition gives that $a_t(h_t) = \bar{\pi}_t$ and $b_t = \underline{\pi}_t$.

And since $\hat{z}_t(h^x_t, s_t) = 0$, $\forall s_t$, by optimality of the strategy in equilibrium, it is equivalent to the following inequalities:

$$\underline{\pi}^{(1)}_t \leq a_t(h_t) = \bar{\pi}_t \tag{3.3.4}$$

$$\bar{\pi}^{(1)}_t \geq b_t(h_t) = \underline{\pi}_t \tag{3.3.5}$$

$$\underline{\pi}^{(0)}_t \leq a_t(h_t) = \bar{\pi}_t \tag{3.3.6}$$

$$\bar{\pi}^{(0)}_t \geq b_t(h_t) = \underline{\pi}_t \tag{3.3.7}$$

Since that $\bar{\pi}^{(1)}_t > \bar{\pi}_t > \underline{\pi}_t$ and $\underline{\pi}^{(0)}_t < \underline{\pi}_t < \bar{\pi}_t$ by informativeness of the signal, i.e. $q > 1/2$, inequalities (3.3.5) and (3.3.6) hold for sure. So we only need that inequalities (3.3.4) and (3.3.7) hold, and it turns out that both of them are equivalent to (3.3.3).

(If part.) Given the inequality (3.3.3), we shall see that it cannot be the case that $\hat{z}_t(h_t^x, 1) = 1$ and $\hat{z}_t(h_t^x, 0) = -1$, which yields

$$\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} \leq \frac{q}{1-q}\frac{1-q+\gamma}{q+\gamma}, \text{ with } q > 1/2,$$

where $\gamma = \frac{1-\eta}{\eta} > 0$, contradictory to the inequality (3.3.3). Then, by Lemma 3.3.2, it must be the case that $\hat{z}_t(h_t^x, s_t) = 0$, $\forall s_t$. Then, we have that, for all v,

$$\Pr(z_t|h_t, v) = \begin{cases} \frac{1-\eta}{3} & \text{if } z_t \neq 0 \\ \frac{1-\eta}{3} + \eta & \text{if } z_t = 0 \end{cases}.$$

Note that, therefore,

$$\begin{aligned}\Pi_{t+1}(h_t, a_t, b_t, z_t) &= \left\{\frac{\pi \Pr(z_t|h_t, 1)}{\pi \Pr(z_t|h_t, 1) + (1-\pi)\Pr(z_t|h_t, 0)} : \pi \in \Pi_t(h_t)\right\} \\ &= \{\pi : \pi \in \Pi_t(h_t)\} \\ &= \Pi_t(h_t)\end{aligned}$$

for all a_t, b_t, and z_t. And then the rest of the proof is done by induction. $\square$

How can we interpret the inequality (3.3.3)? Note that $\frac{\pi}{1-\pi}$ is a strictly increasing transformation of π, and the left-hand side of the inequality is the ratio between $\frac{\bar{\pi}_t}{1-\bar{\pi}_t}$ and $\frac{\underline{\pi}_t}{1-\underline{\pi}_t}$, which captures how different the best and worst beliefs are and provides a concrete measurement of the amount of ambiguity in the common set of beliefs. In the right-hand side, $\frac{q}{1-q} \in (1, \infty)$ also measures the accuracy of the signals. Therefore, the inequality essentially says that the amount of ambiguity of the public beliefs is greater than the accuracy of the signals. And it is sufficient and necessary for having informational cascades in equilibrium. Moreover, once the informational cascade starts, it will last forever, and the bid-ask spread will not converge. The bid-ask spread purely depends on how different the best and worst beliefs are since it is exactly the difference of them. Such an equilibrium with informational cascades exists. We can think of such a game with Π_1 satisfy the inequality (3.3.3), then informational cascade start at period 1 and last forever, which is stated in the following corollary.

Corollary 3.3.1. *If Π_1 satisfies inequality (3.3.3), then there exists an equilibrium where there is an informational cascade in each period, moreover, the bid-ask spread never vanishes.*

Next we consider when it is possible to have the equilibrium resembling that of no ambiguity in the previous section. That is, the informed trader buys the asset if she receives a good signal and sells if receiving a bad signal. Such condition given in the following proposition is again very intuitive, it states that the amount of ambiguity of the public beliefs should be small enough to induce the informed trader to separate themselves in the trading with different signals received.

Proposition 3.3.3. *In an equilibrium, given h_t in period t,*

$$\hat{z}_t(h_t^x, 1) = 1 \text{ and } \hat{z}_t(h_t^x, 0) = -1$$

if and only if

$$\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} \leq \frac{q}{1-q}\frac{1-q+\gamma}{q+\gamma} \tag{3.3.8}$$

where $\gamma = \frac{1-\eta}{3\eta}$.

Proof. (Only if part.) Given h_t, if $\hat{z}_t(h_t^x, 1) = 1$ and $\hat{z}_t(h_t^x, 0) = -1$, then market maker's conditions yields that

$$a_t(h_t) = \frac{\bar{\pi}_t\gamma + \bar{\pi}_t q}{\gamma + \bar{\pi}_t q + (1-\bar{\pi}_t)(1-q)} \tag{3.3.9}$$

$$b_t(h_t) = \frac{\underline{\pi}_t\gamma + \underline{\pi}_t(1-q)}{\gamma + \underline{\pi}_t(1-q) - (1-\underline{\pi}_t)q}. \tag{3.3.10}$$

Note that

$$\hat{z}_t(h_t^x, 1) = 1 \implies \min \Pi_t^{(1)} = \underline{\pi}_t^{(1)} \geq a_t(h_t), \tag{3.3.11}$$

$$\hat{z}_t(h_t^x, 0) = -1 \implies \max \Pi_t^{(0)} = \bar{\pi}_t^{(0)} \leq b_t(h_t). \tag{3.3.12}$$

It turns out that both inequalities (3.3.11) and (3.3.12) are equivalent to:

$$\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} \leq \frac{q}{1-q}\frac{1-q+\gamma}{q+\gamma}.$$

(If part.) For this direction, from Lemma 3.3.2, suppose, for contradiction, it is not $\hat{z}_t(h_t^x, 1) = 1$ and $\hat{z}_t(h_t^x, 0) = -1$. Then it must be that $\hat{z}_t(h_t^x, s) = 0$ for $s \in \{0, 1\}$, and thus, by Proposition 3.3.2, we will have

$$\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} > \frac{q}{1-q},$$

which is contradictory to the inequality 3.3.8 since

$$\frac{1-q+\gamma}{q+\gamma} < 1$$

implied by $q > 1/2$ and $\gamma > 0$. $\square$

Look at the inequality (3.3.8). The left-hand side is the same as the inequality (3.3.3), and the right-hand side is the informativeness of the signal divided by $\frac{q+\gamma}{1-q+\gamma} \in$

$\left(\frac{q}{1-q}, 1\right)$, which is in fact the informativeness of a trading order. Therefore, we have $\frac{q}{1-q}\frac{1-q+\gamma}{q+\gamma} \in \left(1, \frac{q}{1-q}\right)$, and this condition just restricts that the amount of ambiguity should not exceed the ratio of the informativeness of a signal and that of a trading order. It also makes clear that when there is no ambiguity, the left-hand side is equal to one, so the inequality always holds, and thus informed traders are separating in the equilibrium.

Moreover, the inequality (3.3.8) tells us something about the role of the noise traders in the market. When the fraction of the noise traders is higher (γ is higher), it is more likely for the informed traders to trade according to their private information in equilibrium because it is easier for them to hide their order behind the noisy orders. This indicates that noise traders are actually facilitating financial trading.

We still leave some important questions open at the moment, especially, how the beliefs evolve over time and whether the equilibrium always exists with pure trading strategies. For the first question, we refer to a later result, Proposition 3.3.6. In fact, the key measurement of ambiguity, $\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}$, is constant in any equilibrium of a trading game with non-ambiguous private signals. Therefore, by checking whether $\frac{\bar{\pi}_1(1-\underline{\pi}_1)}{\underline{\pi}_1(1-\bar{\pi}_1)}$ is greater than $q/(1-q)$ or less than $\frac{q}{1-q}\frac{1-q+\gamma}{q+\gamma}$, we are able to know that there will be equilibrium where there are informational cascades or not.

However, in contrast to the results of trading games without ambiguity, a trading game with ambiguity does not always have an equilibrium in pure trading strategies. Inspecting the inequalities (3.3.3) and (3.3.8), they are not forming an exhaustive description of the possible set of common beliefs. Since

$$\frac{q}{1-q}\frac{1-q+\gamma}{q+\gamma} < \frac{q}{1-q}$$

because $q > 1/2$ and $\gamma = \frac{1-\eta}{3\eta} > 0$. However, for some arbitrary set Π_t of the common beliefs, it can be the case that

$$\frac{q}{1-q}\frac{1-q+\gamma}{q+\gamma} < \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} < \frac{q}{1-q}. \tag{3.3.13}$$

If there is an equilibrium in pure trading strategies, then there cannot be any history h_t in equilibrium such that $\Pi_t(h_t)$ satisfies the inequality (3.3.13).

Lemma 3.3.3. *There does not exist any history h_t in equilibrium with pure trading strategies such that the inequality (3.3.13) hold for $\Pi_t(h_t)$.*

Proof. Given an equilibrium, suppose, for contradiction, there is such a history h_t in equilibrium. By Lemma 3.3.2, we will have either (i) $\hat{z}_t(h_t^x, 1) = \hat{z}_t(h_t^x, 0) = 0$ or (ii) $\hat{z}_t(h_t^x, 1) = 1$ and $\hat{z}_t(h_t^x, 0) = -1$. But in each case, as shown by Proposition 3.3.2 and 3.3.3, $\Pi_t(h_t)$ will not satisfy inequality (3.3.13). This is the contradiction. $\square$

As a corollary of Lemma 3.3.3, there cannot be any equilibrium with the inequality hold for Π_1. If there is such an equilibrium, it will contradict to the lemma. So we have the following proposition without repeating the proof.

Proposition 3.3.4. *Suppose that Π_1 satisfies inequality (3.3.13), then there is no equilibrium in pure trading strategy.*

However, in Section 3.3.3, when traders play mixed trading strategies, there does exist an equilibrium.

3.3.3 The Binary Model with Mixed Trading Strategy

As showed in the general model, allowing mixed strategy brings back the convexity of the strategy space, therefore, from the point of view of a fixed point argument for existence of equilibrium, it is sufficient to guarantee the existence of equilibrium for any set of common beliefs. Moreover, we will use the results in Thereom 3.2.3 to fully characterize the equilibrium in the binary model with ambiguous priors.

Let us think of an environment similar to that in Section 3.3.2 except for that we allow traders to play mixed strategies. To remind readers, as in the general model, a *mixed strategy* for informed agent t, is $\sigma_t : H_t \times \mathbb{R}_+^2 \times S \to \Delta$, where $\sigma_t = (\zeta_t, \iota_t)$ and

$$\zeta_t(h_t, a_t, b_t, s) = \Pr\left(z_t = 1 | h_t, a_t, b_t, s\right),$$

$$\iota_t(h_t, a_t, b_t, s) = \Pr\left(z_t = -1 | h_t, a_t, b_t, s\right).$$

Then, this just a special case of the general model, so by applying Theorem 3.2.1, there must be an equilibrium for this game. This equilibrium strategy is unique and characterize this equilibrium in the binary model. Following Theorem 3.2.3, we have the following lemma.

Lemma 3.3.4. *It must be case that*

$$\zeta_t(h_t^x, 0) = 0,\ \zeta_t(h_t^x, 1) \geq 0 \text{ and}$$

$$\iota_t(h_t^x, 0) \geq 0,\ \iota_t(h_t^x, 1) = 0$$

for any history h_t in equilibrium.

Proof. For history h_t in equilibrium, there can be either an informational cascade or not. From Corollary 3.2.3, if there is informational cascade in equilibrium, then it must be the case that $\zeta_t(h_t^x, 0) = \zeta_t(h_t^x, 1) = 0$ and $\iota_t(h_t^x, 0)\iota_t(h_t^x, 1) = 0$. Otherwise, we must have $s_t^* = 1$ and $s_{t*} = 0$. Hence, in this case, will have $\zeta_t(h_t^x, 0) = 0$, $\zeta_t(h_t^x, 1) > 0$ and $\iota_t(h_t^x, 0) > 0$, $\iota_t(h_t^x, 1) = 0$. □

Given this lemma, we can focus on the analysis of possible values of $\zeta_t(h_t^x, 1)$ and $\iota_t(h_t^x, 0)$ only. From the earlier section, we have known the case where $\zeta_t(h_t^x, 1) = 0$ and $\iota_t(h_t^x, 0) = 0$ is in fact the informational cascade, and the case where $\zeta_t(h_t^x, 1) = 1$

and $\iota_t(h_t^x, 0) = 1$ is the separating equilibrium. Moreover, as a matter of fact, thanks to the symmetry assumption, $\zeta_t(h_t^x, 1)$ and $\iota_t(h_t^x, 0)$ are also equal when inequality (3.3.13) holds.

More importantly, if inequality (3.3.13) holds in equilibrium, then the market maker will post the prices that make the informed traders just indifferent between trading or not and the informed traders must play strictly mixed trading strategies. Let's focus on the intuition of setting the ask price, $a_t = \underline{\pi}_t^{(1)}$. Given the inequality (3.3.13), it implies that $\underline{\pi}_t^{(1)} > \bar{\pi}_t$. So if the market maker sets the ask price slightly above $\underline{\pi}_t^{(1)}$, then no informed trader will come to buy the asset, then it makes a strictly positive profit from a "buy". But the failure of zero min-expected profit implies the pressure of competition will drive the price down, and hence the market maker will not set an ask price above $\underline{\pi}_t^{(1)}$. More subtly, it is impossible to set ask price below $\underline{\pi}_t^{(1)}$ either because that will contradict with the zero min-expected profit as well. Suppose so, from Lemma 3.2.1, to avoid expected loss, the market maker has to set ask price to be at least greater than $\underline{\pi}_t^{(1)}$ when inequality (3.3.13) holds. Along with the analysis before, this gives us a complete picture of how should be the pricing and trading decisions look like in any history in equilibrium for different sets of common beliefs, which are summarized in the following proposition.

Proposition 3.3.5. *In any equilibrium, given history h_t in equilibrium and $\Pi_t(h_t)$, then we have the following:*

(i) there exists a unique $\sigma(h_t) \in (0, 1)$ such that

$$\zeta_t(h_t^x, 1) = \iota_t(h_t^x, 0) = \sigma(h_t),$$
$$a_t(h_t) = \underline{\pi}_t^{(1)},$$
$$b_t(h_t) = \bar{\pi}_t^{(0)},$$

if and only if

$$\frac{q}{1-q}\frac{1-q+\gamma}{q+\gamma} < \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} < \frac{q}{1-q};$$

(ii)

$$\zeta_t(h_t^x, 1) = \iota_t(h_t^x, 0) = 0,$$
$$a_t(h_t) = \bar{\pi}_t,$$
$$b_t(h_t) = \underline{\pi}_t,$$

if and only if

$$\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} \geq \frac{q}{1-q};$$

(iii)

$$\begin{aligned}\zeta_t(h_t^x, 1) &= \iota_t(h_t^x, 0) = 1\\ a_t(h_t) &= \frac{\bar{\pi}_t\gamma + \bar{\pi}_t q}{\gamma + \bar{\pi}_t q + (1-\bar{\pi}_t)(1-q)},\\ b_t(h_t) &= \frac{\underline{\pi}_t\gamma + \underline{\pi}_t(1-q)}{\gamma + \underline{\pi}_t(1-q) - (1-\underline{\pi}_t)q},\end{aligned}$$

if and only if

$$\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} \leq \frac{q}{1-q}\frac{1-q+\gamma}{q+\gamma}.$$

Proof. In the Appendix to Chapter 3. □

Belief updating

Proposition 3.3.5 only prescribes the trading and pricing strategies. However, to fully characterize the equilibrium, we need to also specify the consistent beliefs. Moreover, understanding the evolution of beliefs enables us to answer questions like whether it is possible to switch from non-cascade to an informational cascade.

Specifically, the public belief, or the set of common beliefs, is updated through the observable to the public, or the information contained in the public history, i.e. the ask, a_t, and bid, b_t, and trading orders, z_t. Given the equilibrium strategies, the conditional probabilities of a "buy" at history h_t is

$$\begin{aligned}\Pr(z_t = 1|h_t, v = 1) &= \frac{1-\eta}{3} + \eta q \sigma(h_t),\\ \Pr(z_t = 1|h_t, v = 0) &= \frac{1-\eta}{3} + \eta(1-q)\sigma(h_t).\end{aligned}$$

and the conditional probabilities of a "sell" at history h_t is

$$\begin{aligned}\Pr(z_t = -1|h_t, v = 1) &= \frac{1-\eta}{3} + \eta(1-q)\sigma(h_t),\\ \Pr(z_t = -1|h_t, v = 0) &= \frac{1-\eta}{3} + \eta q \sigma(h_t).\end{aligned}$$

Define the *likelihood ratio* function $l_t : Z \to \mathbb{R}_+$ by $l_t(z_t) = \frac{\Pr(z_t|h_t, v=1)}{\Pr(z_t|h_t, v=0)}$. Then, given the equilibrium strategy, by Bayes rule,

$$\frac{\pi_{t+1}}{1-\pi_{t+1}} = \frac{\pi_t}{1-\pi_t} l_t(z_t),$$

for all $\pi_t \in \Pi_t(h_t)$, specifically for $\bar{\pi}_t$ and $\underline{\pi}_t$ as well. Therefore, $\Pi_{t+1}(h_t^x, z_t)$ can be characterized by

$$\left\{ \pi_{t+1} \in (0,1) |\ \exists \pi_t \in \Pi_t(h_t) : \frac{\pi_{t+1}}{1-\pi_{t+1}} = \frac{\pi_t}{1-\pi_t} l_t(z_t) \right\}, \tag{3.3.14}$$

and we have the following proposition that tells that $\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}$, the measurement of ambiguity, is constant in the market over time.

Proposition 3.3.6. *Given an equilibrium,*

$$\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} = \frac{\bar{\pi}_1(1-\underline{\pi}_1)}{\underline{\pi}_1(1-\bar{\pi}_1)}$$

for all $t = 1, 2, \ldots$ and history h_t in equilibrium.

Proof. First, fix an equilibrium and h_t. Note that by the information structure, for an arbitrary $z_t \in Z$, use l_t as short for $l_t(z_t)$.

$$\begin{aligned}
\bar{\pi}_{t+1} &= \max \Pi_{t+1}(h_t^x, z_t) \\
&= \max_{\pi_t \in \Pi_t(h_{t+1})} \frac{\pi_t \Pr(z_t|h_t, v=1)}{\pi_t \Pr(z_t|h_t, v=1) + (1-\pi_t)\Pr(z_t|h_t, v=0)} \\
&= \frac{\bar{\pi}_t \Pr(z_t|h_t, v=1)}{\bar{\pi}_t \Pr(z_t|h_t, v=1) + (1-\bar{\pi}_t)\Pr(z_t|h_t, v=0)}
\end{aligned}$$

and therefore,

$$\frac{\bar{\pi}_{t+1}}{1-\bar{\pi}_{t+1}} = \frac{\bar{\pi}_t}{1-\bar{\pi}_t} l_t.$$

and similarly,

$$\frac{\underline{\pi}_{t+1}}{1-\underline{\pi}_{t+1}} = \frac{\underline{\pi}_t}{1-\underline{\pi}_t} l_t.$$

Hence,

$$\frac{\bar{\pi}_{t+1}}{1-\bar{\pi}_{t+1}} \frac{1-\underline{\pi}_{t+1}}{\underline{\pi}_{t+1}} = \frac{\bar{\pi}_t}{1-\bar{\pi}_t} \frac{1-\underline{\pi}_t}{\underline{\pi}_t},$$

and rest of the proof done by mathematical induction. □

The Characterization of equilibrium

The Proposition 3.3.6 shows that $\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}$ is constant along the equilibrium path. Therefore, informed agent's decision in period t, $t > 1$, will be exactly the same as that in period 1. Then we can construct the equilibrium as follow. For the pricing strategies, we first compute $\frac{\bar{\pi}_1(1-\underline{\pi}_1)}{\underline{\pi}_1(1-\bar{\pi}_1)}$. If inequality (3.3.8) holds, then prices are set as equations (3.3.9) and (3.3.10) as functions of $\Pi_t(h_t)$ in each period t, and given these prices the informed trader will buy fore sure with a good signal and sell for sure with a bad signal. If it is the case that inequality (3.3.13) holds, we set price as $a_t(h_t) = \underline{\pi}_t^{(1)}$ and $b_t(h_t) = \bar{\pi}_t^{(0)}$. Consequently, the informed traders are indifferent between not to trade and to buy with good signal and indifferent between not to trade and to sell with a bad signal. So the informed traders choose the probability of not trading to make the market maker's zero min-expected profit conditions hold. Finally, if inequality (3.3.3) holds, then prices are set as $a_t(h_t) = \bar{\pi}_t$ and $b_t(h_t) = \underline{\pi}_t$, and the optimal decision for informed traders is never to trade. In these three cases, consistent belief are constructed in the way that described as before, specifically, in the last case it turns out that $\Pi_t = \Pi_1$ for all t. We can verify that this construction satisfies all the conditions to be an equilibrium, and it also characterizes the unique equilibrium strategies. In fact, in each case there cannot be other ways to price the asset to satisfy all the equilibrium conditions, so this construction has to be what an equilibrium outcome looks like. We summarize the results in the following proposition.

Proposition 3.3.7. *Given a trading game under ambiguity, there can be one of the following equilibria:*

(i) a pure separating equilibrium, where, for all $t \geq 1$ and h_t in equilibrium,

$$\begin{aligned}
\zeta_t(h_t^x, 1) &= \iota_t(h_t^x, 0) = 1 \\
\zeta_t(h_t^x, 0) &= \iota_t(h_t^x, 1) = 0 \\
a_t(h_t) &= \frac{\bar{\pi}_t \gamma + \bar{\pi}_t q}{\gamma + \bar{\pi}_t q + (1 - \bar{\pi}_t)(1 - q)} \\
b_t(h_t) &= \frac{\underline{\pi}_t \gamma + \underline{\pi}_t (1 - q)}{\gamma + \underline{\pi}_t (1 - q) - (1 - \underline{\pi}_t) q}
\end{aligned}$$

and beliefs evolve as (3.3.14) if and only if inequality (3.3.8) holds for Π_1;

(ii) a mixed separating equilibrium, where there exists a unique $\sigma \in (0,1)$ such that, for all $t \geq 1$ and h_t in equilibrium,

$$\zeta_t(h_t^x, 1) = \iota_t(h_t^x, 0) = \sigma$$

$$\zeta_t(h_t^x, 0) = \iota_t(h_t^x, 1) = 0$$

$$a_t(h_t) = \underline{\pi}_t^{(1)} \text{ and } b_t(h_t) = \bar{\pi}_t^{(0)}$$

and beliefs evolve as (3.3.14) if and only if inequality (3.3.13) holds for Π_1;

(iii) an informational cascade equilibrium, where, for all $t \geq 1$ and h_t in equilibrium,

$$\zeta_t(h_t^x, 1) = \iota_t(h_t^x, 0) = 0$$

$$\zeta_t(h_t^x, 0) = \iota_t(h_t^x, 1) = 0$$

$$a_t(h_t) = \bar{\pi}_1 \text{ and } b_t(h_t) = \underline{\pi}_0$$

and $\Pi_t = \Pi_1$ for every $t \geq 1$ if and only if inequality (3.3.3) holds for Π_1.

Proof. In the Appendix to Chapter 3. □

Here, we can extend the Proposition 4 of Glosten and Milgrom (1985) and the Proposition 4 Avery and Zemsky (1998) in an ambiguous environment but only when there is no informational cascade. If $\frac{\bar{\pi}_1(1-\underline{\pi}_1)}{\underline{\pi}_1(1-\bar{\pi}_1)} < \frac{q}{1-q}$, then the equilibrium prices a_t and b_t will converge to v and ask-bid spread will converge to 0 almost surely. Otherwise, in the case of informational cascade, $\frac{\bar{\pi}_1(1-\underline{\pi}_1)}{\underline{\pi}_1(1-\bar{\pi}_1)} \geq \frac{q}{1-q}$, the equilibrium prices will be constant, and ask-bid spread is strictly positive and never converges to 0.

Figure 3.3.2-3.3.4 illustrate three different types of equilibria. We set $v = 1$, $\eta = 0.5$, and $q = 0.6$ in all examples and choose different $\bar{\pi}_1$ and $\underline{\pi}_1$ so that conditions for different types of equilibria hold. We can see that in the pure separating equilibrium, Figure 3.3.2, it is not much different from the case without ambiguity. In the mixed

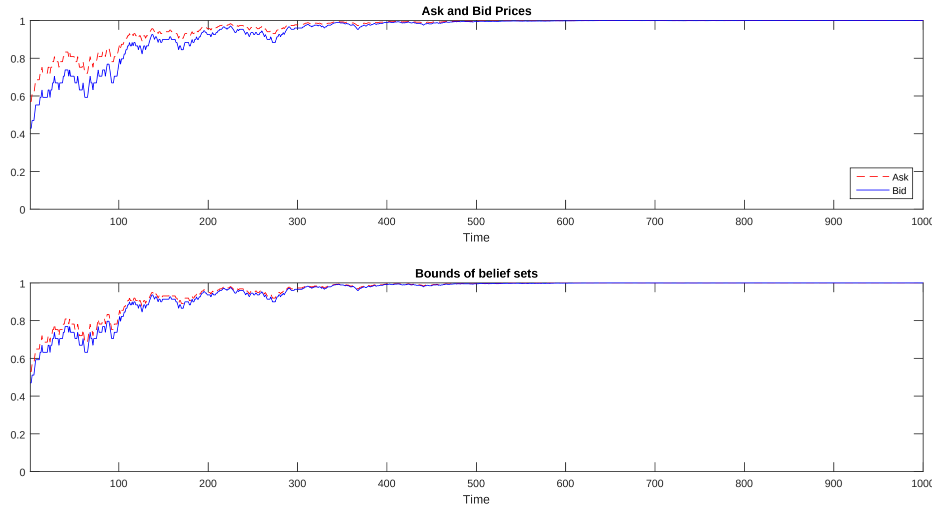

Figure 3.3.2: Binary model with ambiguous initial priors ($\underline{\pi}_1 = 0.47$,$\bar{\pi}_1 = 0.53$)

separating equilibrium, Figure 3.3.3, although convergence is predicted by theory, it may take extraordinary long time. In this sample path, there is no any sign of convergence even after 1000 rounds of trading. Lastly, in the informational cascade equilibrium, Figure 3.3.4, everything is just as predicted to be a straight line.

Comparative Statics of Ambiguity

How does ambiguity in market itself affect the ask-bid spread? Consider two public beliefs at some history h_t, $\Pi_t(h_t)$ and $\Pi_t'(h_t)$. $\Pi_t'(h_t)$ is said to be *more ambiguous* than $\Pi_t(h_t)$ if $\Pi_t'(h_t) \subset \Pi_t(h_t)$. In fact, the more ambiguous relation defined here is consistent with the measurement of ambiguity $\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}$ mentioned beforehand. $\Pi_t'(h_t)$ is more ambiguous than $\Pi_t(h_t)$ implies that $\frac{\bar{\pi}_t'(1-\underline{\pi}_t')}{\underline{\pi}_t'(1-\bar{\pi}_t')} > \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}$. Moreover, the following result shows that, loosely speaking, the ask-bid spread is increasing in ambiguity.

Theorem 3.3.1. *Let $(a_t(h_t), b_t(h_t))$ and $(a_t'(h_t), b_t'(h_t))$ be the equilibrium ask and bid prices corresponding to the beliefs $\Pi_t(h_t)$ and $\Pi_t'(h_t)$, respectively. If $\Pi_t'(h_t)$ is more ambiguous than $\Pi_t(h_t)$, then $a_t'(h_t) - b_t'(h_t) > a_t(h_t) - b_t(h_t)$.*

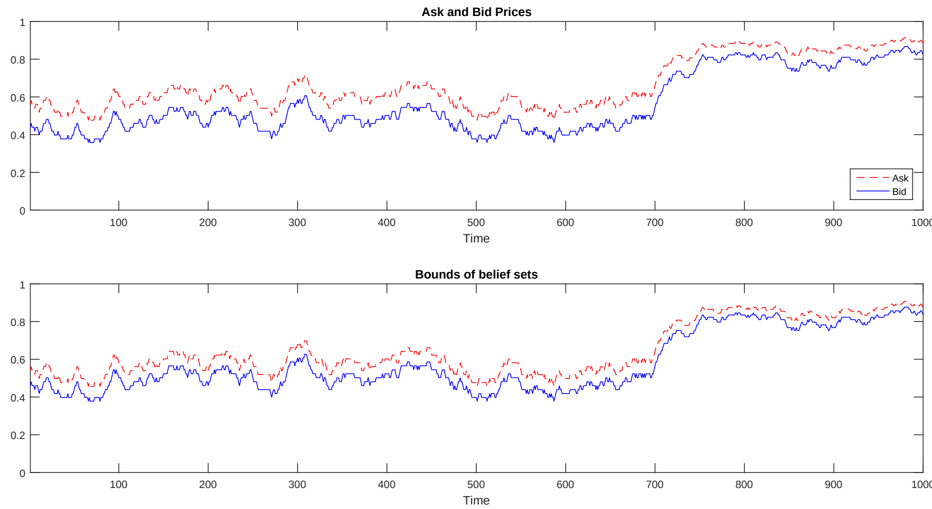

Figure 3.3.3: Binary model with ambiguous initial priors ($\underline{\pi}_1 = 0.46$,$\bar{\pi}_1 = 0.54$)

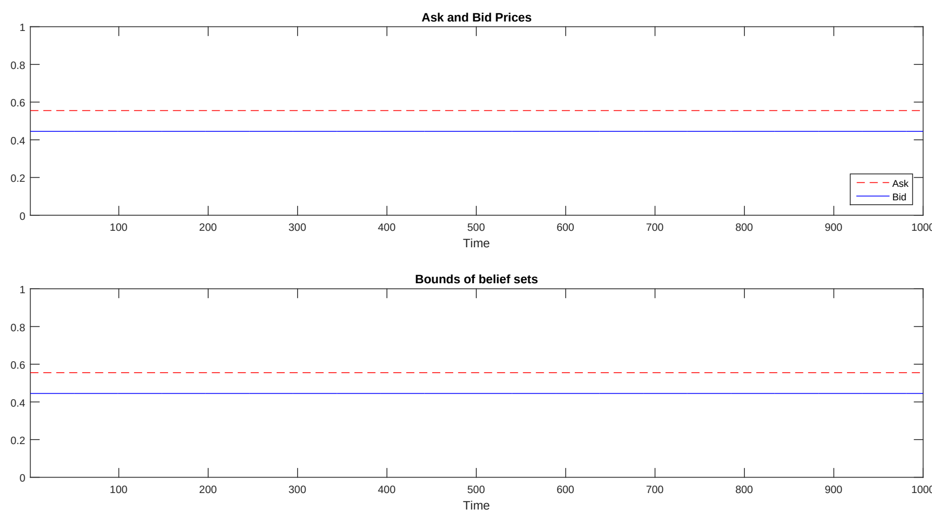

Figure 3.3.4: Binary model with ambiguous initial priors ($\underline{\pi}_1 = 0.445$,$\bar{\pi}_1 = 0.555$)

Proof. In the Appendix to Chapter 3. □

Intuitively, when the market maker is ambiguity averse, it will push up the ask price and press down the bid price to compensate its aversion to ambiguity. When there is more ambiguity in the market, this force of enlarging ask-bid spread is stronger. Hence, uncertainty aversion reinforces the problem of adverse selection in the sequential trading, and more uncertainty induces larger ask-bid spread.

3.4 Extensions of the Binary Model

3.4.1 Ambiguity in the Private Signals and the Asymptotics of Equilibrium

In this section, we further consider the binary trading game with ambiguous private signals. In Section 3.3, the equilibrium in binary model with ambiguous prior is not so interesting in the sense that there cannot be any switch of different types of equilibrium: either there are informational cascades from fist period on, or there is no informational cascade forever because the key measurement of ambiguity is constant in the equilibrium path. However, when ambiguous private signals are introduced in the model, the ambiguity of the common beliefs will grow because the ambiguity of private signals is added into it whenever the trading orders reveals the information of the private signals imperfectly.

Formally assume that the signal structure is the one in Example 3.2.2. It is like the previous sections that the binary signal is symmetric and characterized by some $q > 1/2$, however the traders are not sure about the value of q. They consider that $Q = [\underline{q}, \bar{q}]$ with $\underline{q} < \bar{q}$. First, we extend some notations: $\pi^{(s,q)}$ denote the updated belief, or the posterior, given prior π and signal s and under the conditional distribution q, and let

$$\pi^{(s,q)} = \frac{\pi \Pr(s|1;q)}{\pi \Pr(s|1;q) + (1-\pi)\Pr(s|0;q)}$$

where

$$\Pr(s|1;q) = \begin{cases} q & \text{if } s = 1 \\ 1-q & \text{if } s = 0 \end{cases},$$

and

$$\Pi_t^{(s)} := \left\{ \pi_t^{(s,q)} | \, \pi_t \in \Pi_t(h_t), q \in \{\underline{q}, \bar{q}\} \right\}.$$

Here also consider the mixed trading strategies to guarantee the existence of equilibrium. In an equilibrium, depending on the set $\Pi_t(h_t)$ of common beliefs. there can be three situations, which looks just similar to the cases without ambiguity in signals, except that the thresholds are changed to capture this new ambiguity.

First, if there is an informational cascade, i.e. $\zeta_t(h_t^x, 1) = \iota_t(h_t^x, 0) = 0$, then prices would still be $a_t(h_t) = \bar{\pi}_t$ and $b_t(h_t) = \underline{\pi}_t$. To make the trading decision optimal, we shall have $a_t(h_t) \geq \underline{\pi}_t^{(1,\underline{q})}$ and $b_t(h_t) \leq \bar{\pi}_t^{(0,\bar{q})}$. And both of them is equivalent to

$$\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} \geq \frac{\underline{q}}{1-\underline{q}}. \tag{3.4.1}$$

Secondly, if it is the pure separating case, i.e. $\zeta_t(h_t^x, 1) = \iota_t(h_t^x, 0) = 1$, then the prices would be

$$a_t(h_t) = \frac{\bar{\pi}_t\gamma + \bar{\pi}_t\bar{q}}{\gamma + \bar{\pi}_t\bar{q} + (1-\bar{\pi}_t)(1-\bar{q})}, \text{ and}$$
$$b_t(h_t) = \frac{\underline{\pi}_t\gamma + \underline{\pi}_t(1-\bar{q})}{\gamma + \underline{\pi}_t(1-\bar{q}) - (1-\underline{\pi}_t)\bar{q}}.$$

To make this trading decision optimal, we shall have $a_t(h_t) \leq \underline{\pi}_t^{(1,\underline{q})}$ and $b_t(h_t) \geq \bar{\pi}_t^{(0,\bar{q})}$. Both of them is equivalent to

$$\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} \leq \frac{\underline{q}}{1-\underline{q}}\frac{1-\bar{q}+\gamma}{\bar{q}+\gamma}. \tag{3.4.2}$$

Thirdly, for the mixed separating case, we will have $\zeta_t(h_t^x, 1) = \iota_t(h_t^x, 0) = \sigma_t \in (0,1)$, then prices would be $a_t(h_t) = \underline{\pi}_t^{(1,\underline{q})}$ and $b_t(h_t) = \bar{\pi}_t^{(0,\bar{q})}$. To make the trading decision optimal, we shall have

$$\begin{aligned} \frac{\bar{\pi}_t\gamma/\sigma_t + \bar{\pi}_t\bar{q}}{\gamma/\sigma_t + \bar{\pi}_t\bar{q} + (1-\bar{\pi}_t)(1-\bar{q})} &= \frac{\underline{\pi}_t\underline{q}}{\underline{\pi}_t\underline{q} + (1-\underline{\pi}_t)(1-\underline{q})} \\ \Longrightarrow \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} &= \frac{\underline{q}}{1-\underline{q}}\frac{1-\bar{q}+\gamma/\sigma_t}{\bar{q}+\gamma/\sigma_t} \in \left(\frac{\underline{q}}{1-\underline{q}}\frac{1-\bar{q}+\gamma}{\bar{q}+\gamma}, \frac{\underline{q}}{1-\underline{q}}\right) \end{aligned}$$

However, the above just tells us about what the pricing and trading decision would be in particular history h_t in equilibrium. The next proposition shows that the set of common beliefs now evolves in a way that the measurement of ambiguity is increasing over time.

Proposition 3.4.1. *In equilibrium,*

$$\frac{\bar{\pi}_{t+1}(1-\underline{\pi}_{t+1})}{\underline{\pi}_{t+1}(1-\bar{\pi}_{t+1})} \geq \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}, \ \forall t = 1, 2, \dots,$$

with the inequality being strict if $z_t \neq 0$ *and* $\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} < \frac{\underline{q}}{1-\underline{q}}$.

Proof. The probability of a buy or a sell is ambiguous now, for example, $\Pr(z_t = 1|h_t, v = 1)$ can be either $\frac{1-\eta}{3} + \bar{q}\sigma$ or $\frac{1-\eta}{3} + \underline{q}\sigma$, where $\sigma \in [0,1]$. or equivalently, we have multiple likelihood ratio function $l_t(\cdot; q) : S \to \mathbb{R}_+$, depending on the value of q, defined by

$$l_t(z_t; q) = \frac{\Pr(z_t|h_t, v=1; q)}{\Pr(z_t|h_t, v=0; q)}$$

where the conditional probabilities are defined as before, but we emphasize its dependence on q. If $z_t = 0$, then $l_t(z_t; q) \equiv 1$. Otherwise, l_t is strictly increasing in q when $z_t = 1$ and decreasing in q when $z_t = -1$.

Moreover, given some h_t in equilibrium and $z_t = 1$

$$\begin{aligned}\frac{\bar{\pi}_{t+1}}{1-\bar{\pi}_{t+1}} &= \max_{\pi\in\Pi_t, q\in\{\underline{q},\bar{q}\}} \frac{\pi}{1-\pi} l_t(1;q) \\ &= \frac{\bar{\pi}_t}{1-\bar{\pi}_t} l_t(1;\bar{q})\end{aligned}$$

and

$$\begin{aligned}\frac{\underline{\pi}_{t+1}}{1-\underline{\pi}_{t+1}} &= \min_{\pi\in\Pi_t, q\in\{\underline{q},\bar{q}\}} \frac{\pi}{1-\pi} l_t(1;q) \\ &= \frac{\underline{\pi}_t}{1-\underline{\pi}_t} l_t(1;\underline{q}),\end{aligned}$$

and hence,

$$\begin{aligned}\frac{\bar{\pi}_{t+1}}{1-\bar{\pi}_{t+1}} \frac{1-\underline{\pi}_{t+1}}{\underline{\pi}_{t+1}} &= \frac{\bar{\pi}_t}{1-\bar{\pi}_t} \frac{1-\underline{\pi}_t}{\underline{\pi}_t} \frac{l_t(1;\bar{q})}{l_t(1;\underline{q})} \\ &> \frac{\bar{\pi}_t}{1-\bar{\pi}_t} \frac{1-\underline{\pi}_t}{\underline{\pi}_t}\end{aligned}$$

and similarly when $z_t = -1$,

$$\begin{aligned}\frac{\bar{\pi}_{t+1}}{1-\bar{\pi}_{t+1}} \frac{1-\underline{\pi}_{t+1}}{\underline{\pi}_{t+1}} &= \frac{\bar{\pi}_t}{1-\bar{\pi}_t} \frac{1-\underline{\pi}_t}{\underline{\pi}_t} \frac{l_t(-1;\underline{q})}{l_t(-1;\bar{q})} \\ &> \frac{\bar{\pi}_t}{1-\bar{\pi}_t} \frac{1-\underline{\pi}_t}{\underline{\pi}_t}.\end{aligned}$$

□

As said before, when private information is revealed, albeit not perfectly, the ambiguity accompanied by the private signal is also injected into the public beliefs. Notice that if inequality (3.4.1) holds at period t, the probability of trading $\sigma_t = 1$

is independent of specific values of the set $\Pi_t(h_t)$, and thus $\frac{l_t(z_t;\underline{q})}{l_t(z_t;\bar{q})} > 1$ is fixed, if $z_t \neq 0$. therefore, heuristically, when periods is longer enough, it is almost surely that $\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}$ will pass over the threshold, $\frac{\underline{q}}{1-\underline{q}}\frac{1-\bar{q}+\gamma}{\bar{q}+\gamma}$, of playing mixed trading strategy in equilibrium. However, when $\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}$ is between $\frac{\underline{q}}{1-\underline{q}}\frac{1-\bar{q}+\gamma}{\bar{q}+\gamma}$ and $\frac{\underline{q}}{1-\underline{q}}$, the probability of trading σ_t depends endogenous on the specific value of the measurement of ambiguity, and it converges to 0 as $\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}$ increases towards $\frac{\underline{q}}{1-\underline{q}}$, and the likelihood ratio $\frac{l_t(z_t;\underline{q})}{l_t(z_t;\bar{q})}$ converges to 1 for $z_t \neq 0$. Therefore, at the first glance, it is not clear whether $\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}$ will pass the threshold $\frac{\underline{q}}{1-\underline{q}}$ to enter the stage of informational cascade. We conclude that this possibility is null, showed in the next proposition, with calculation of σ_t explicitly, nevertheless, $\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}$ converges to $\frac{\underline{q}}{1-\underline{q}}$ almost surely in equilibrium.

Proposition 3.4.2. *With ambiguous signal, if*

$$\frac{\bar{\pi}_1(1-\underline{\pi}_1)}{\underline{\pi}_1(1-\bar{\pi}_1)} < \frac{\underline{q}}{1-\underline{q}},$$

then, in equilibrium,

$$\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} < \frac{\underline{q}}{1-\underline{q}}$$

for all $t = 1, 2, \ldots$*; however,* $\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}$ *converges to* $\frac{\underline{q}}{1-\underline{q}}$ *as* $t \to \infty$ *almost surely.*

Proof. In the Appendix to Chapter 3. □

Although an informational cascade cannot occur along the equilibrium path, the convergence result is still useful to show that the public beliefs of the market can be arbitrarily close to the situation where there is an informational cascade. Therefore, the trading decisions can be arbitrarily close to that of an informational cascade when there are sufficient long periods to accumulate ambiguous signals. To formalize this idea, we define an ε-informational cascade by that the difference of trading decisions across different types of agents is very small. Formally, Let $\|\cdot\|$ be the sup-norm.

Definition 3.4.1. Given the strategy profile (x, σ) and $\varepsilon > 0$, there is an *ε-informational cascade* after history h_t if

$$\left\|\sigma_t\left(h_t^x, s_t\right) - \sigma_t\left(h_t^x, s_t'\right)\right\| < \varepsilon$$

for all $s_t, s_t' \in S$.

Then, we can state this asymptotic result as the following theorem. We can conjecture that this will be also true in the general model since the general idea should also apply there.

Theorem 3.4.1. *If the signals are ambiguous in the binary model, then there will be an ε-informational cascade as $t \to \infty$ almost surely for any $\varepsilon > 0$.*

Proof. Fix $\varepsilon > 0$. We can see that there is an ε-informational cascade if and only if

$$\left|\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} - \frac{\underline{q}}{1-\underline{q}}\right| < \delta_\varepsilon$$

for some $\delta_\varepsilon > 0$. By Proposition 3.4.2, this event happens as $t \to \infty$ almost surely. $\square$

At the end of this part, there is another sample paths, Figure 3.4.1, attached to illustrate the dynamics of prices and beliefs in the model with ambiguous signals. Due the fact that signal is ambiguous and not too accurate ($\underline{q} = 0.6$ and $\bar{q} = 0.61$), we can see that after several hundreds rounds of trading, it then looks like an informational cascade. In fact, we can think of that the market enter some ε-informational cascade with this ε smaller than the machine epsilon in the computer.

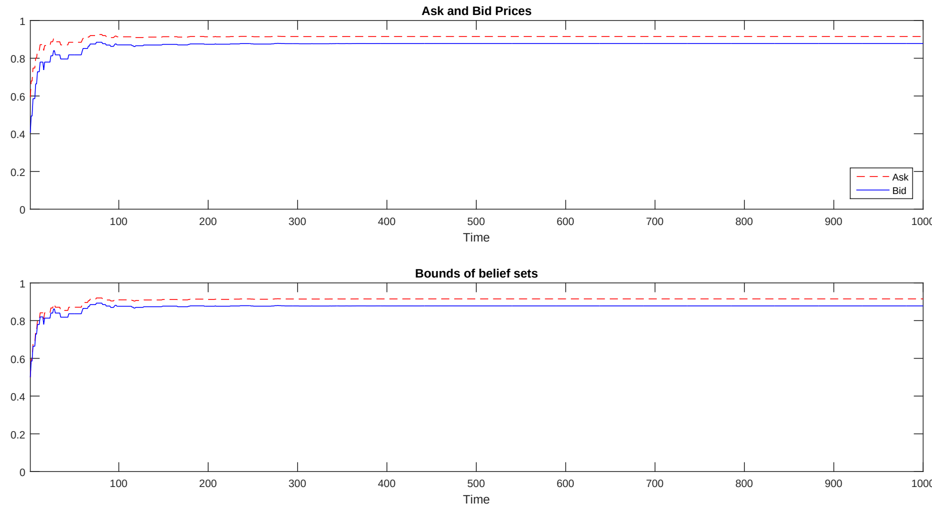

Figure 3.4.1: Binary model with ambiguous signals ($v = 1$, $\Pi_1 = \{0.5\}$, $q = 0.6$,$\eta = 0.8$)

3.4.2 Asymmetric Signals: No News is Bad (Good) News

One consequence of the existence of equilibrium with mixed trading strategies is that it is possible for informed traders not to submit any order to the market maker. However, when the signals are symmetric as in Section 3.3, no trade at period t, i.e. $z_t = 0$, is still not informative in the sense that the set of conditional probabilities of v conditional on $z_t = 0$ is the same as the public belief $\Pi_t(h_t)$. To see this, for any belief $\pi \in \Pi_t(h_t)$, we denote the posterior $\Pr^\pi(v = 1|z_t = 0)$ *the updated belief conditional on observing* $z_t = 0$ *with prior belief* π. Let $\hat{\zeta}$ and $\hat{\iota}$ be the equilibrium trading strategy, so that $\zeta_t(h_t^x, 1) = \iota_t(h_t^x, 0) = \sigma \in [0, 1]$ by Proposition 3.3.7. Then,

$$\begin{aligned}\Pr(z = 0|v = 1) &= \frac{1-\eta}{3} + \eta\left[(1 - \zeta_t(h_t^x, 1))q + (1 - \iota_t(h_t^x, 0))(1-q)\right] \\ &= \frac{1-\eta}{3} + \eta(1-\sigma) = \Pr(z = 0|v = 0),\end{aligned}$$

which implies that $\Pr^{\pi}(v = 1|z = 0) = \pi$ for all $\pi \in \Pi_t(h_t)$. Hence, $\Pi_{t+1}(h_t^x, z_t = 0) = \Pi_t(h_t)$.

Observe that $\zeta_t(h_t^x, 1) = \iota_t(h_t^x, 0)$ in every possible equilibrium, and this is primarily driven by the fact that the signals are symmetric. However, if the signals are asymmetric, then the no-trade itself can be informative especially when $\zeta_t(h_t^x, 1) \neq \iota_t(h_t^x, 0)$. Mowever, $\zeta_t(h_t^x, 1) \neq \iota_t(h_t^x, 0)$ is only possible when there is ambiguity in the market. Let's consider a model similar to that in Section 3.3, but assume that binary signals are aymmetric,

$$q_1 = \Pr(s = 1|v = 1) > \Pr(s = 0|v = 0) = q_0.$$

Then the following proposition shows that informed traders respond to good and bad signals asymmetrically under certain condition and thus no-trade would be informative in the sense that the set of conditional probabilities of v conditional on $z_t = 0$ is different from the public belief $\Pi_t(h_t)$.

Proposition 3.4.3. *In the binary model with asymmetric signals, if*

$$\frac{q_1}{1 - q_1} \leq \frac{\bar{\pi}_t(1 - \underline{\pi}_t)}{\underline{\pi}_t(1 - \bar{\pi}_t)} \leq \frac{q_0}{1 - q_0}\frac{1 - q_0 + \gamma}{q_0 + \gamma}, \tag{3.4.3}$$

then $0 \leq \iota_t(h_t^x, 0) < \zeta_t(h_t^x, 1) \leq 1$ *and* $\Pr^{\pi}(v = 1|z = 0) < \pi$ *for all* $\pi \in \Pi_t(h_t)$.

Note that since $\Pr^{\pi}(v = 1|z = 0) < \pi$ for all $\pi \in \Pi_t(h_t)$, it must be the case that $\Pi_{t+1}(h_t^x, z_t = 0) \neq \Pi_t(h_t)$ because $\Pr^{\underline{\pi}_t}(v = 1|z = 0) < \underline{\pi}_t$.

We omit the formal proof Proposition 3.4.3, but refer to the following discussion of the equilibrium in the binary model with asymmetric signals. First, if there is an informational cascade in h_t, then

$$\frac{\bar{\pi}_t(1 - \underline{\pi}_t)}{\underline{\pi}_t(1 - \bar{\pi}_t)} \geq \max\left\{\frac{q_1}{1 - q_1}, \frac{q_0}{1 - q_0}\right\} = \frac{q_1}{1 - q_1}.$$

In the informational cascade, all the buy or sell orders comes from the noisy traders and all the informed traders choose to not to trade, no activity is informative. On the other hand, when

$$\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} \leq \frac{q_0}{1-q_0}\frac{1-q_0+\gamma}{q_0+\gamma},$$

informed traders with a good signal will buy the asset and traders with a bad signal will sell, i.e. $\zeta_t(h_t^x, 1) = \iota_t(h_t^x, 0) = 1$, in equilibrium, which also nests the case where there is no ambiguity. In this case, since $\zeta_t(h_t^x, 1) = \iota_t(h_t^x, 0)$, no-trade is still not informative.

Now we consider some specific cases where inequality 3.4.3 can hold. First, if

$$\frac{q_0}{1-q_0} \leq \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} < \frac{q_1}{1-q_1}, \tag{3.4.4}$$

then it is possible to have that $\zeta_t(h_t^x, 1) > 0$ but $\iota_t(h_t^x, 0) = 0$, i.e. only informed traders with a good signal will buy the asset, none of the orders will come from informed traders with a bad signal. When

$$\frac{q_0}{1-q_0}\frac{1-q_0+\gamma}{q_0+\gamma} < \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} \leq \frac{q_1}{1-q_1}\frac{1-q_1+\gamma}{q_1+\gamma}, \tag{3.4.5}$$

we will have $\zeta_t(h_t^x, 1) = 1$ but $\iota_t(h_t^x, 0) < 1$, i.e. all the informed traders with a good signal will buy the asset, while informed traders with a bad signal will sell the asset only with some probability less than one. However, there can be two other cases, when

$$\frac{q_1}{1-q_1}\frac{1-q_1+\gamma}{q_1+\gamma} < \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} < \frac{q_0}{1-q_0},$$

informed traders optimally choose that $0 < \iota_t(h_t^x, 0) < \zeta_t(h_t^x, 1) < 1$. Otherwise, if it is such that

$$\frac{q_0}{1-q_0} \leq \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} \leq \frac{q_1}{1-q_1}\frac{1-q_1+\gamma}{q_1+\gamma},$$

then both inequality (3.4.4) and (3.4.5) hold, and informed traders optimally choose that $\zeta_t(h_t^x, 1) = 1$ and $\iota_t(h_t^x, 0) = 0$, i.e. all the informed traders with a good signal will buy the asset, but none of the traders with a bad signal will sell. Hence, in equilibrium, we always have $\zeta_t(h_t^x, 1) > \iota_t(h_t^x, 0)$ since the good signal is more accurate than the bad signal. Moreover, for the last case, it is interesting because informed trader will only buy, and no sell will come from the informed traders. Consequently, no-trade lead to a decrease in the valuation of the asset, but a sell order does not.

In these cases, no-trade, $z_t = 0$, becomes informative here since it hints that it is more likely to be resulted from some informed trader with a bad signal because the bad signals are less informative. Therefore, when no-trade occurs, the valuation of the asset will decrease in the market: No news is bad news when the bad signal is less informative. Conversely, if the good signal is less informative, i.e. $q_1 < q_0$, then no news is good news. The valuation of the asset will increase when no-trade occurs because it is more likely that informative traders with a good signal choose not to trade.

3.5 Conclusion

When the public belief is ambiguous and agents are ambiguity averse, informed traders and market maker behave differently from situation where there is no ambiguity. Informed traders may be reluctant to trade due to their aversion to uncertainty, which can lead to informational cascade. The fundamental reason for such informational cascade is different from that of Bikhchandani et al. (1992), where informational cascade relies on the difference between the public belief and fixed investment cost. But, in the asset trading environment, the cost of investment is adjusting and reflecting the public beliefs. Therefore, the informational cascade here is in fact a *status quo* bias as a consequence of aversion to uncertainty. And such phenomenon is robustly attributed to ambiguity aversion, by which I mean that the results will be largely preserved even if we further generalize the model, for example, having a continuum set of the value of the asset, and/or allowing trading volume to be positive real numbers.

The market maker tend to ask for high selling prices and bid for low buying prices of the asset if they are ambiguity averse and the public belief is ambiguous. The ambiguous the market maker, the larger the ask-bid spread. The need for the market maker to be compensated for their aversion to ambiguity widens the ask-bid spread and worsens the problem of adverse selection.

One of the reasons that we can think of the common belief is ambiguous can be traced to the fact that the private signals are ambiguous. When the informed traders trade relying on ambiguous private information, although there is new information assimilated into the market, the ambiguity accompanied that piece of information is also injected to the public beliefs.

The fact that ambiguity keeps the informed traders away from trading has significant consequences. It makes the market fail to incorporate every information flowing around effectively if the informed trader is not trading. When there are much ambiguous private information flowing around in the market over extensively long periods, the beliefs would become too ambiguous for informed traders to trade sufficiently frequently. It will either take super long periods to make belief to converge to some rational expectations, or beliefs are just stay there and trading activities are purely random. Therefore, the results reasonably cast doubts on the efficient market hypothesis and rational expectation assumptions, especially when there is much (Knightian) uncertainty in the market.

Since the informed traders can play mixed trading strategies in equilibrium under ambiguity, no-trade can also be informative because it is less likely for the informed trader with a specific signal to place any order. As shown in the paper, if the bad signal is less informative than the good signal, then informed traders with a bad signal are less likely to trade. No news, i.e. no trade in the market, is bad news because it is more likely resulted from a bad signal. But conversely, if the good signal is less informative than the bad signal, then informed traders with a good signal are less likely to trader and thus no news is good news.

Reference

Anat R Admati. A noisy rational expectations equilibrium for multi-asset securities markets. *Econometrica: Journal of the Econometric Society*, pages 629–657, 1985.

Werner Antweiler and Murray Z Frank. Is all that talk just noise? the information content of internet stock message boards. *The Journal of Finance*, 59(3):1259–1294, 2004.

Efstathios Avdis. Information tradeoffs in dynamic financial markets. *Journal of Financial Economics*, 122(3):568–584, 2016.

Christopher Avery and Peter Zemsky. Multidimensional uncertainty and herd behavior in financial markets. *American economic review*, pages 724–748, 1998.

Kerry Back and Shmuel Baruch. Information in securities markets: Kyle meets glosten and milgrom. *Econometrica*, 72(2):433–465, 2004.

Kerry Back, C Henry Cao, and Gregory A Willard. Imperfect competition among informed traders. *The journal of finance*, 55(5):2117–2155, 2000.

Malcolm Baker and Jeffrey Wurgler. Investor sentiment and the cross-section of stock returns. *The Journal of Finance*, 61(4):1645–1680, 2006.

Abhijit V Banerjee. A simple model of herd behavior. *The Quarterly Journal of Economics*, pages 797–817, 1992.

Snehal Banerjee and Bradyn M Breon-Drish. Dynamic information acquisition and strategic trading. 2016.

Gadi Barlevy and Pietro Veronesi. Information acquisition in financial markets. *The Review of Economic Studies*, 67(1):79–90, 2000.

Gadi Barlevy, Pietro Veronesi, et al. Information acquisition in financial markets: a correction. Technical report, Federal Reserve Bank of Chicago, 2007.

Sushil Bikhchandani, David Hirshleifer, and Ivo Welch. A theory of fads, fashion, custom, and cultural change as informational cascades. *Journal of political Economy*, pages 992–1026, 1992.

Christophe Chamley. *Rational herds: Economic models of social learning*. Cambridge University Press, 2004.

Varadarajan V Chari and Patrick J Kehoe. Financial crises as herds: overturning the critiques. *Journal of Economic Theory*, 119(1):128–150, 2004.

Marco Cipriani and Antonio Guarino. Herd behavior and contagion in financial markets. *The BE Journal of Theoretical Economics*, 8(1), 2008.

Russell Cooper and Andrew John. Coordinating coordination failures in keynesian models. *The Quarterly Journal of Economics*, 103(3):441–463, 1988.

Inés Couso, Serafın Moral, and Peter Walley. Examples of independence for imprecise probabilities. 1999.

Jean-Paul Decamps and Stefano Lovo. Informational cascades with endogenous prices: The role of risk aversion. *Journal of Mathematical Economics*, 42(1):109–120, 2006.

Jean-Paul Décamps and Thomas Mariotti. Investment timing and learning externalities. *Journal of Economic Theory*, 118(1):80–102, 2004.

Douglas W Diamond and Robert E Verrecchia. Information aggregation in a noisy rational expectations economy. *Journal of Financial Economics*, 9(3):221–235, 1981.

Marco Antonio Guimaraes Dias and Jose Paulo Teixeira. Continuous-time option games: Review of models and extensions. *Multinational Finance Journal*, 14(3-4): 219–254, 2010.

Avinash K Dixit and Robert S Pindyck. *Investment under uncertainty*. Princeton university press, 1994.

Joseph L Doob. Stochastic processes depending on a continuous parameter. *Transactions of the American Mathematical Society*, 42(1):107–140, 1937.

James Dow and Sergio Ribeiro da Costa Werlang. Uncertainty aversion, risk aversion, and the optimal choice of portfolio. *Econometrica: Journal of the Econometric Society*, pages 197–204, 1992.

Prajit K Dutta and Aldo Rustichini. A theory of stopping time games with applications to product innovations and asset sales. *Economic Theory*, 3(4): 743–763, 1993.

Daniel Ellsberg. Risk, ambiguity, and the savage axioms. *The quarterly journal of economics*, pages 643–669, 1961.

Larry G Epstein and Martin Schneider. Recursive multiple-priors. *Journal of Economic Theory*, 113(1):1–31, 2003.

Larry G Epstein and Tan Wang. Intertemporal asset pricing under knightian uncertainty. *Econometrica: Journal of the Econometric Society*, pages 283–322, 1994.

Larry G Epstein, Martin Schneider, et al. Ambiguity and asset markets. *Annual Review of Financial Economics*, 2(1):315–346, 2010.

Pablo Fajgelbaum, Edouard Schaal, and Mathieu Taschereau-Dumouchel. Uncertainty traps. *Quarterly Journal of Economics*, forthcoming.

Eugene F Fama and Kenneth R French. The cross-section of expected stock returns. *the Journal of Finance*, 47(2):427–465, 1992.

Maryam Farboodi and Laura Veldkamp. Financial technology, unpredictability and illiquidity in the long run. 2016.

F Douglas Foster and S Viswanathan. Strategic trading when agents forecast the forecasts of others. *The Journal of Finance*, 51(4):1437–1478, 1996.

Drew Fudenberg and Jean Tirole. Preemption and rent equalization in the adoption of new technology. *The Review of Economic Studies*, 52(3):383–401, 1985.

Jayant Vivek Ganguli and Liyan Yang. Complementarities, multiplicity, and supply information. *Journal of the European Economic Association*, 7(1):90–115, 2009.

Itzhak Gilboa and David Schmeidler. Maxmin expected utility with non-unique prior. *Journal of mathematical economics*, 18(2):141–153, 1989.

Lawrence R Glosten and Paul R Milgrom. Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. *Journal of financial economics*, 14(1):71–100, 1985.

Steven R Grenadier. Option exercise games: An application to the equilibrium investment strategies of firms. *The Review of Financial Studies*, 15(3):691–721, 2002.

Sanford J Grossman and Joseph E Stiglitz. On the impossibility of informationally efficient markets. *The American economic review*, 70(3):393–408, 1980.

Martin F Hellwig. On the aggregation of information in competitive markets. *Journal of economic theory*, 22(3):477–498, 1980.

Arvid OI Hoffmann and Hersh Shefrin. Technical analysis and individual investors. *Journal of Economic Behavior & Organization*, 107:487–511, 2014.

Craig W Holden and Avanidhar Subrahmanyam. Long-lived private information and imperfect competition. *The Journal of Finance*, 47(1):247–270, 1992.

KJM Huisman and PM Kort. Effects of strategic interactions on the option value of waiting. *CentER Discussion Paper*, 1999, 1999.

Kenneth L Judd. The law of large numbers with a continuum of iid random variables. *Journal of Economic theory*, 35(1):19–25, 1985.

Marcin Kacperczyk, Stijn Van Nieuwerburgh, and Laura Veldkamp. A rational theory of mutual funds' attention allocation. *Econometrica*, 84(2):571–626, 2016.

Eugene Kandel and Neil D Pearson. Differential interpretation of public signals and trade in speculative markets. *Journal of Political Economy*, 103(4):831–872, 1995.

A Ya Khinchin. *Mathematical foundations of information theory*. Courier Corporation, 2013.

Albert S Kyle. Continuous auctions and insider trading. *Econometrica: Journal of the Econometric Society*, pages 1315–1335, 1985.

Bart M Lambrecht. The impact of debt financing on entry and exit in a duopoly. *The Review of Financial Studies*, 14(3):765–804, 2001.

David F Larcker and Thomas Lys. An empirical analysis of the incentives to engage in costly information acquisition: The case of risk arbitrage. *Journal of Financial Economics*, 18(1):111–126, 1987.

In Ho Lee. Market crashes and informational avalanches. *The Review of Economic Studies*, 65(4):741–759, 1998.

John Lintner. The aggregation of investor's diverse judgments and preferences in purely competitive security markets. *Journal of financial and quantitative analysis*, 4(4):347–400, 1969.

Andreu Mas-Colell. On a theorem of schmeidler. *Journal of Mathematical Economics*, 13(3):201–206, 1984.

Haim Mendelson and Tunay I Tunca. Strategic trading, liquidity, and information acquisition. *The Review of Financial Studies*, 17(2):295–337, 2003.

Lukas Menkhoff. The use of technical analysis by fund managers: International evidence. *Journal of Banking & Finance*, 34(11):2573–2586, 2010.

Jordi Mondria. Portfolio choice, attention allocation, and price comovement. *Journal of Economic Theory*, 145(5):1837–1864, 2010.

Sujoy Mukerji and Jean-Marc Tallon. Ambiguity aversion and incompleteness of financial markets. *The Review of Economic Studies*, 68(4):883–904, 2001.

Pauli Murto. Exit in duopoly under uncertainty. *RAND Journal of Economics*, pages 111–127, 2004.

Andreas Park and Hamid Sabourian. Herding and contrarian behavior in financial markets. *Econometrica*, 79(4):973–1026, 2011.

Lin Peng and Wei Xiong. Investor attention, overconfidence and category learning. *Journal of Financial Economics*, 80(3):563–602, 2006.

Cesaltina Pacheco Pires. A rule for updating ambiguous beliefs. *Theory and Decision*, 53(2):137–152, 2002.

Kali P Rath. A direct proof of the existence of pure strategy equilibria in games with a continuum of players. *Economic Theory*, 2(3):427–433, 1992.

David Schmeidler. Equilibrium points of nonatomic games. *Journal of statistical Physics*, 7(4):295–300, 1973.

Claude Elwood Shannon. A mathematical theory of communication. *The Bell System Technical Journal*, 27:379–423, 1948.

Christopher A Sims. Implications of rational inattention. *Journal of monetary Economics*, 50(3):665–690, 2003.

Frank Rafael Smets. *Essays on foreign direct investment.* PhD thesis, Yale University, 1994.

Lones Smith and Peter Sørensen. Pathological outcomes of observational learning. *Econometrica*, 68(2):371–398, 2000.

Yeneng Sun. The exact law of large numbers via fubini extension and characterization of insurable risks. *Journal of Economic Theory*, 126(1):31–69, 2006.

Mark P Taylor and Helen Allen. The use of technical analysis in the foreign exchange market. *Journal of international Money and Finance*, 11(3):304–314, 1992.

Jacco JJ Thijssen, Kuno JM Huisman, and Peter M Kort. The effects of information on strategic investment and welfare. *Economic Theory*, 28(2):399–424, 2006.

Stijn Van Nieuwerburgh and Laura Veldkamp. Information immobility and the home bias puzzle. *The Journal of Finance*, 64(3):1187–1215, 2009.

Stijn Van Nieuwerburgh and Laura Veldkamp. Information acquisition and under-diversification. *The Review of Economic Studies*, 77(2):779–805, 2010.

Robert E Verrecchia. Information acquisition in a noisy rational expectations economy. *Econometrica: Journal of the Econometric Society*, pages 1415–1430, 1982.

Xavier Vives. *Information and learning in markets: the impact of market microstructure.* Princeton University Press, 2010.

Appendix A

Appendix to Chapter 1

A.1 Proofs in Chapter 1

Proof of Theorem 1.2.1.

Proof. The proof is done by a guess-and-verify procedure. First, we guess that $p = \frac{1}{R}(A + B\theta - C\zeta)$ and solve the undetermined constants A, B, and C.

We have seen that in the main text, for agent a, it is straightforward that $q_a(s_a, z_a, p) = \lambda_a^{-1}\pi'_a(\theta'_a - pR)$. Therefore, we have

$$\int_0^1 \lambda_a^{-1}\pi'_a(\theta'_a - pR)\,da = \zeta$$

or

$$\int_0^1 \lambda_a^{-1}\pi'_a\theta'_a da - \zeta = pR\hat{\pi}', \tag{A.1.1}$$

where $\hat{\pi}' = \int_0^1 \lambda_a^{-1}\pi'_a da$. Then, the LHS of equation (A.1.1) is

$$\begin{aligned}
\text{LHS} &= \int_0^1 \lambda_a^{-1}\left[\pi_{\theta,a}\bar{\theta} + \sigma_{\theta,a}(\theta + \epsilon_i) + D^2\rho_{z,a}\left(\theta - \left(\zeta - \rho_{z,a}^{-1}\left(\pi_\zeta\bar{\zeta} + \sigma_{\zeta,a}(\zeta + \xi_i)\right)\right)/D\right)\right]da - \zeta \\
&= \hat{\Pi}_\theta\bar{\theta} + D\hat{\Pi}_\zeta\bar{\zeta} + \left(\hat{\Sigma}_\theta + D^2\hat{\Pi}_z\right)\theta - \left(D\hat{\Pi}_z - D\hat{\Sigma}_\zeta + 1\right)\zeta
\end{aligned}$$

where $\hat{\Pi}_\theta = \int_0^1 \lambda_a^{-1}\pi_{\theta,a}da$, $\hat{\Pi}_\zeta = \int_0^1 \lambda_a^{-1}\pi_{\zeta,a}da$, $\hat{\Sigma}_\theta = \int_0^1 \lambda_a^{-1}\sigma_{\theta,a}da$, $\hat{\Pi}_\zeta = \int_0^1 \lambda_a^{-1}\sigma_{\zeta,a}da$, and $\hat{\Pi}_z = \int_0^1 \lambda_a^{-1}\pi_{z,a}da = \hat{\Pi}_\zeta + \hat{\Sigma}_\zeta$. The RHS is $\hat{\pi}'(A + B\theta - C\zeta)$, and hence, we have

$$\begin{aligned}
\frac{\hat{\Sigma}_\theta + D^2\hat{\Pi}_z}{D\hat{\Pi}_z - D\hat{\Sigma}_\zeta + 1} &= D \\
\Longleftrightarrow \hat{\Sigma}_\zeta D^2 - D + \hat{\Sigma}_\theta &= 0.
\end{aligned}$$

so we can solve D and a real solution exists if and only if $\Delta = 1 - 4\hat{\Sigma}_\zeta\hat{\Sigma}_\theta \geq 0$. Moreover, by Vieta's Theorem, there must be a strictly positive root when $\hat{\Sigma}_\zeta > 0$. When $\hat{\Sigma}_\zeta = 0$, there always exists a positive root.

Then, with some $D > 0$,

$$\begin{aligned}\pi'_a &= \pi_{\theta,a} + \sigma_{\theta,a} + D^2 \pi_{z,a} \\ &= \pi_{\theta,a} + \sigma_{\theta,a} + D^2 \left(\pi_{\zeta,a} + \sigma_{\zeta,a}\right), \\ \hat{\pi}' &= \int_0^1 \lambda_a^{-1} \pi'_a da = \hat{\Pi}_\theta + \hat{\Sigma}_\theta + D^2 \left(\hat{\Pi}_\zeta + \hat{\Sigma}_\zeta\right),\end{aligned}$$

and thus, with the fact that $\hat{\Pi}_z = \hat{\Pi}_\zeta + \hat{\Sigma}_\zeta$,

$$\begin{aligned}C &= \left(D\hat{\Pi}_\zeta + 1\right) / \hat{\pi}' \\ B &= \left[\hat{\Sigma}_\theta + D^2 \left(\hat{\Pi}_\zeta + \hat{\Sigma}_\zeta\right)\right] / \hat{\pi}' \\ A &= \left(\hat{\Pi}_\theta \bar{\theta} + D\hat{\Pi}_\zeta \bar{\zeta}\right) / \hat{\pi}'\end{aligned}$$

Then, we have expressions of $x = A/C$ and $c = C/R$.

Finally, we can verify that, when $\hat{\Sigma}_\theta = 0$, $D = 0$ along with

$$\begin{aligned}c &= \frac{1}{R\hat{\Pi}_\theta}, \\ x &= \hat{\Pi}_\theta \bar{\theta}\end{aligned}$$

can also constitute an equilibrium price $p = c(x - \zeta)$ that clears the market. □

Proof of Proposition 2.3.1.

Proof. Fix some asset market equilibrium D. As a corollary of Proposition 1.2.1, from equation (A.1.1), we have

$$E\left[\int_0^1 \lambda_a^{-1} \pi'_a \theta'_a da - \zeta\right] = E\left[pR\pi'\right]$$

and switching the integral and expectation operator and using the martingale property, i.e. $E\left[\theta'_a\right] = \bar{\theta}$,

$$\text{LHS} = \int_0^1 \lambda_a^{-1} \pi'_a E\left[\theta'_a\right] da - \bar{\zeta} = \hat{\pi}' \bar{\theta} - \bar{\zeta},$$

and denote $\bar{p} = E\left[\tilde{p}\right]$,

$$\text{RHS} = \pi' \bar{p} R,$$

and hence,

$$\bar{\theta} - \bar{p}R = \bar{\zeta}/\pi' \tag{A.1.2}$$

The objective function $U_a^D\left(\sigma_{\theta,a},\sigma_{\zeta,a};\hat{\Sigma}_\theta,\hat{\Sigma}_\zeta\right)$ can be substituted with

$$U_a^D=-E\left[\exp\left[-\frac{\pi_a'}{2}\left(\theta_a'-pR\right)^2\right]\right]$$

by plugging in the optimal portfolio in the equilibrium and removing the constant term. Note that $(\theta_a'-pR)\sim N(m_a,V_a)$ where

$$\begin{aligned}m_a&=E_a\left[\theta_a'-pR\right]\\&=\bar{\theta}-\bar{p}R\\&=\bar{\zeta}/\hat{\pi}'\end{aligned}$$

by equation (A.1.2) and use $Var[x]=E[Var[x|y]]+Var[E[x|y]]$, we have

$$\begin{aligned}V_a&=Var_a\left[E_a\left[\theta-pR|s_a,z_a,p\right]\right]\\&=Var_a\left[\theta-pR\right]-E_a\left[Var_a\left[\theta-pR|s_a,z_a,p\right]\right]\\&=(1-B)^2/\pi_{\theta,a}+C^2/\pi_{\zeta,a}-1/\pi_a'\end{aligned}$$

Denote $\Theta_a=V_a^{-1/2}\left(\theta_a'-pR\right)$ the Sharpe ratio and it has a normal distribution with unit variance, so $\Theta_a^2\sim\chi(1,V_a^{-1/2}m_a)$. Use the MGF of a non-central chi square distribution, i.e.

$$E\left[e^{t\tilde{x}}\right]=(1-2t)^{-k/2}\exp\left(\frac{\lambda t}{1-2t}\right)$$

where $\tilde{x}\sim\chi(k,\lambda)$, so we have

$$\begin{aligned}U_a^D&=-E\left[\exp\left[-\frac{\pi_a'}{2}\left(\theta_a'-pR\right)^2\right]\right]\\&=-E\left[\exp\left(-\frac{\pi_a'V_a}{2}\Theta_a^2\right)\right]\\&=-\left(1+\pi_a'V_a\right)^{-1/2}\exp\left(\frac{-\pi_a'V_a^{1/2}m_i}{2\left(1+\pi_a'V_a\right)}\right)\\&=-(L_a\pi_a')^{-1/2}\exp\left[\frac{-\lambda\bar{\zeta}\sqrt{L_a-\pi_a'}}{2L_a\hat{\pi}'}\right]\end{aligned}$$

where $L_a=(1-B)^2/\pi_{\theta,a}+C^2/\pi_{\zeta,a}$. Note that U_a^D depends on $\sigma_{\theta,a}$ and $\sigma_{\zeta,a}$ only through π_a' and it is strictly increasing in π_a':

$$\begin{aligned}\frac{dU_a^D}{d\pi_a'}=&\frac{1}{2}L_a^{-1/2}\pi_a'\exp\left[\frac{-\lambda\bar{\zeta}\sqrt{L_a-\pi_a'}}{2L_a\hat{\pi}'}\right]\\&+\frac{1}{\pi_a'}(L_a\pi_a')^{-1/2}\exp\left[\frac{-\lambda\bar{\zeta}\sqrt{L_a-\pi'}}{2L_a\hat{\pi}'}\right]\frac{\lambda\bar{\zeta}}{2L_a\hat{\pi}'}\frac{1}{2\sqrt{L_a-\pi_a'}}>0.\end{aligned}$$

Hence, we can replace the objective with $\pi'_a = \pi_{\theta,a} + \sigma_{\theta,a} + D^2 \left(\pi_{\zeta,a} + \sigma_{\zeta,a}\right)$. $\square$

Proof of Proposition 1.3.2.

Proof. Suppose that $\left(\sigma_{\theta,a}, \sigma_{\zeta,a}\right)_{a\in[0,1]}$ and $D = 0$ be an overall equilibrium. From Proposition 2.3.1, individual's problem would be $\max_{\left(\sigma_{\theta,a},\sigma_{\zeta,a}\right)\in K_a} \{\pi_{\theta,a} + \sigma_{\theta,a}\}$, hence the only optimal choice is that $\sigma_{\theta,a} = \max\{\sigma_{\theta,a}|(\sigma_{\theta,a}, \sigma_{\zeta,a}) \in K_a\} > 0$ if the agent can learn the payoff. Since the set of agents who can learn the payoff is strictly positive, it then must be the case that $\hat{\Sigma}_\theta > 0$, by Theorem 1.2.1, D has to solve equation (1.2.11). But $D = 0$ cannot be a solution to that equation when $\hat{\Sigma}_\theta > 0$. This reaches a contradiction.

Conversely, suppose that the set of agents who can learn the payoff is measure zero. We can construct an overall equilibrium where $D = 0$. If the agent cannot learn the payoff, they have to choose $\sigma_{\theta,a} = 0$ since $\sigma_a = 0$ for every $(\sigma_{\theta,a}, \sigma_{\zeta,a}) \in K_a$, but they can choose some $\sigma_{\zeta,a} > 0$. If the agents can learn the payoff, then they choose $(\sigma_{\theta,a}, \sigma_{\zeta,a}) \in K_a$ such that $\sigma_{\theta,a} = \max\{\sigma_{\theta,a}|(\sigma_{\theta,a}, \sigma_{\zeta,a}) \in K_a\}$. Since the set of agents who can learn the payoff is measure zero, it is the case that $\hat{\Sigma}_\theta = 0$ and $\hat{\Sigma}_\zeta > 0$. Then, we can verify that $D = 0$ can be supported in this overall equilibrium. $\square$

Proof of Proposition 2.3.2.

Proof. Proof is simply verifying the equilibrium conditions. We will prove (i) only, the proof of (ii) is similar.

Given $\mu^*(\pi_{\theta,a}) = 1$ for all $\pi_{\theta,a} \in \text{supp}F$, $\Sigma_\theta = (H-1)\Pi_\theta$ and $\Sigma_\zeta = 0$; hence, $D = (H-1)\Pi_\theta/\lambda$. So $\mu^*(\pi_{\theta,a}) = 1$ for all $\pi_{\theta,a} \in \text{supp}F$ if and only if $(H-1)\Pi_\theta/\lambda \leq \sqrt{\underline{\pi}_\theta/\pi_\zeta}$. And in fact,

$$\begin{aligned} D =& \frac{(H-1)\Pi_\theta}{\lambda} \leq \sqrt{\frac{\underline{\pi}_\theta}{\pi_\zeta}} \\ \iff& \bar{\pi}_\theta \pi_\zeta \leq \lambda^2/(H-1)^2. \end{aligned}$$

So the conclusion follows. $\square$

Proof of Proposition 2.3.3.

Proof. Suppose that $\mu^*(\bar{\pi}_\theta) = 1$ and $\mu^*(\underline{\pi}_\theta) = 0$ is an equilibrium. Then,

$$\begin{aligned} \Sigma_\theta &= \alpha(H-1)\bar{\pi}_\theta, \\ \Sigma_\zeta &= (1-\alpha)(H-1)\pi_\zeta. \end{aligned}$$

Consequently,

$$
\begin{aligned}
D &= \frac{\lambda \pm \sqrt{\lambda^2 - 4\Sigma_\theta \Sigma_\zeta}}{2\Sigma_\zeta} \\
&= \frac{\lambda \pm \sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2 \bar{\pi}_\theta \pi_\zeta}}{2(1-\alpha)(H-1)\pi_\zeta}.
\end{aligned}
$$

We only need to show that $\sqrt{\frac{\underline{\pi}_\theta}{\pi_\zeta}} \leq D \leq \sqrt{\frac{\bar{\pi}_\theta}{\pi_\zeta}}$ for some D as above.

But we have,

$$
\sqrt{\frac{\underline{\pi}_\theta}{\pi_\zeta}} \leq \frac{\lambda \pm \sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2 \bar{\pi}_\theta \pi_\zeta}}{2(1-\alpha)(H-1)\pi_\zeta} \leq \sqrt{\frac{\bar{\pi}_\theta}{\pi_\zeta}} \tag{A.1.3}
$$

$$
\begin{aligned}
\iff 2(1-\alpha)(H-1)\sqrt{\underline{\pi}_\theta \pi_\zeta} - \lambda &\leq \pm \sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2 \bar{\pi}_\theta \pi_\zeta} \qquad \text{(A.1.4)} \\
&\leq 2(1-\alpha)(H-1)\sqrt{\bar{\pi}_\theta \pi_\zeta} - \lambda.
\end{aligned}
$$

Then, we look at three cases when $\lambda^2 - 4\alpha(1-\alpha)(H-1)^2 \bar{\pi}_\theta \pi_\zeta \geq 0$, which is equivalent to $4\alpha(1-\alpha)\bar{\pi}_\theta \pi_\zeta \leq \lambda^2/(H-1)^2$:

Case (i):

$$
\begin{aligned}
-\sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2 \bar{\pi}_\theta \pi_\zeta} &\leq 2(1-\alpha)(H-1)\sqrt{\underline{\pi}_\theta \pi_\zeta} - \lambda \\
&\leq \sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2 \bar{\pi}_\theta \pi_\zeta} \\
\iff \left[2(1-\alpha)(H-1)\sqrt{\underline{\pi}_\theta \pi_\zeta} - \lambda\right]^2 &\leq \lambda^2 - 4\alpha(1-\alpha)(H-1)^2 \bar{\pi}_\theta \pi_\zeta \\
\iff \frac{\Pi_\theta \pi_\zeta}{\underline{\pi}_\theta} &\leq \frac{\lambda^2}{(H-1)^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2 \bar{\pi}_\theta \pi_\zeta} &\leq 2(1-\alpha)(H-1)\sqrt{\bar{\pi}_\theta \pi_\zeta} - \lambda \\
\implies \lambda^2 - 4\alpha(1-\alpha)(H-1)^2 \bar{\pi}_\theta \pi_\zeta &\leq \left[2(1-\alpha)(H-1)\sqrt{\bar{\pi}_\theta \pi_\zeta} - \lambda\right]^2 \\
\iff \frac{\lambda^2}{(H-1)^2} &\leq \bar{\pi}_\theta \pi_\zeta
\end{aligned}
$$

To summarize, case (i) implies that $\frac{\Pi_\theta \pi_\zeta}{\underline{\pi}_\theta} \leq \frac{\lambda^2}{(H-1)^2} \leq \bar{\pi}_\theta \pi_\zeta$.

Case (ii):

$$
\begin{aligned}
2(1-\alpha)(H-1)\sqrt{\underline{\pi}_\theta \pi_\zeta} - \lambda &\leq -\sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2 \bar{\pi}_\theta \pi_\zeta} \\
\implies \left[2(1-\alpha)(H-1)\sqrt{\underline{\pi}_\theta \pi_\zeta} - \lambda\right]^2 &\geq \lambda^2 - 4\alpha(1-\alpha)(H-1)^2 \bar{\pi}_\theta \pi_\zeta \\
\iff \frac{\Pi_\theta \pi_\zeta}{\underline{\pi}_\theta} &\geq \frac{\lambda^2}{(H-1)^2}
\end{aligned}
$$

and

$$
\begin{aligned}
-\sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2 \bar{\pi}_\theta \pi_\zeta} &\leq 2(1-\alpha)(H-1)\sqrt{\bar{\pi}_\theta \pi_\zeta} - \lambda \\
&\leq \sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2 \bar{\pi}_\theta \pi_\zeta} \\
\iff \left[2(1-\alpha)(H-1)\sqrt{\bar{\pi}_\theta \pi_\zeta} - \lambda\right]^2 &\leq \sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2 \bar{\pi}_\theta \pi_\zeta} \\
\iff \frac{\lambda^2}{(H-1)^2} &\geq \bar{\pi}_\theta \pi_\zeta
\end{aligned}
$$

To summarize, case (ii) implies that $\bar{\pi}_\theta \pi_\zeta \leq \frac{\lambda^2}{(H-1)^2} \leq \frac{\Pi_\theta \pi_\zeta}{\underline{\pi}_\theta}$.

Note that in both case (i) and (ii) $4\alpha(1-\alpha)\bar{\pi}_\theta \pi_\zeta = \min_{0 \leq \underline{\pi}_\theta \leq \bar{\pi}_\theta} \frac{\Pi_\theta \pi_\zeta}{\underline{\pi}_\theta} \leq \frac{\Pi_\theta \pi_\zeta}{\underline{\pi}_\theta}$ and $4\alpha(1-\alpha)\bar{\pi}_\theta \pi_\zeta \leq \max_{\alpha \in [0,1]} 4\alpha(1-\alpha)\bar{\pi}_\theta \pi_\zeta = \bar{\pi}_\theta \pi_\zeta$.

Case (iii):

$$
\begin{aligned}
2(1-\alpha)(H-1)\sqrt{\underline{\pi}_\theta \pi_\zeta} - \lambda &< -\sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2 \bar{\pi}_\theta \pi_\zeta} \\
\implies \left[2(1-\alpha)(H-1)\sqrt{\underline{\pi}_\theta \pi_\zeta} - \lambda\right]^2 &> \lambda^2 - 4\alpha(1-\alpha)(H-1)^2 \bar{\pi}_\theta \pi_\zeta \\
\iff \frac{\lambda^2}{(H-1)^2} &< \frac{\Pi_\theta \pi_\zeta}{\underline{\pi}_\theta}
\end{aligned}
$$

and

$$
\begin{aligned}
\sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2 \bar{\pi}_\theta \pi_\zeta} &< 2(1-\alpha)(H-1)\sqrt{\bar{\pi}_\theta \pi_\zeta} - \lambda \\
\implies \lambda^2 - 4\alpha(1-\alpha)(H-1)^2 \bar{\pi}_\theta \pi_\zeta &< \left[2(1-\alpha)(H-1)\sqrt{\bar{\pi}_\theta \pi_\zeta} - \lambda\right]^2 \\
\iff \frac{\lambda^2}{(H-1)^2} &< \bar{\pi}_\theta \pi_\zeta.
\end{aligned}
$$

To summarize, case (iii) implies that $\frac{\lambda^2}{(H-1)^2} < \min\left\{\frac{\Pi_\theta \pi_\zeta}{\underline{\pi}_\theta}, \bar{\pi}_\theta \pi_\zeta\right\}$. Moreover, $\frac{\lambda^2}{(H-1)^2}$ has a lower bound $4\alpha(1-\alpha)\bar{\pi}_\theta\pi_\zeta$. Moreover, we also know that

$$\begin{aligned}
2(1-\alpha)(H-1)\sqrt{\underline{\pi}_\theta\pi_\zeta} - \lambda < -\sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2\bar{\pi}_\theta\pi_\zeta} \leq 0 \\
\implies \lambda \geq 2(1-\alpha)(H-1)\sqrt{\underline{\pi}_\theta\pi_\zeta} \\
\implies \min\left\{\sqrt{\frac{\Pi_\theta\pi_\zeta}{\underline{\pi}_\theta}}, \sqrt{\bar{\pi}_\theta\pi_\zeta}\right\} > 2(1-\alpha)\sqrt{\underline{\pi}_\theta\pi_\zeta} \\
\implies \alpha > \max\left\{\frac{\underline{\pi}_\theta}{\underline{\pi}_\theta + \bar{\pi}_\theta}, 1 - \frac{1}{2}\sqrt{\frac{\bar{\pi}_\theta}{\underline{\pi}_\theta}}\right\} \\
\implies \alpha > \frac{\underline{\pi}_\theta}{\underline{\pi}_\theta + \bar{\pi}_\theta}
\end{aligned}$$

since $\frac{\underline{\pi}_\theta}{\underline{\pi}_\theta+\bar{\pi}_\theta} > 1 - \frac{1}{2}\sqrt{\frac{\bar{\pi}_\theta}{\underline{\pi}_\theta}} \iff \frac{\underline{\pi}_\theta+\bar{\pi}_\theta}{2} > \sqrt{\underline{\pi}_\theta\bar{\pi}_\theta}$; and

$$\begin{aligned}
0 \leq \sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2\bar{\pi}_\theta\pi_\zeta} < 2(1-\alpha)(H-1)\sqrt{\bar{\pi}_\theta\pi_\zeta} - \lambda \\
\implies \lambda \leq 2(1-\alpha)(H-1)\sqrt{\bar{\pi}_\theta\pi_\zeta} \\
\implies \sqrt{\alpha(1-\alpha)\bar{\pi}_\theta\pi_\zeta} < (1-\alpha)\sqrt{\bar{\pi}_\theta\pi_\zeta} \\
\implies \alpha < 1/2.
\end{aligned}$$

Conversely, let one of the following cases holds: (i) $\frac{\Pi_\theta\pi_\zeta}{\underline{\pi}_\theta} \leq \frac{\lambda^2}{(H-1)^2} \leq \bar{\pi}_\theta\pi_\zeta$, (ii) $\bar{\pi}_\theta\pi_\zeta \leq \frac{\lambda^2}{(H-1)^2} \leq \frac{\Pi_\theta\pi_\zeta}{\underline{\pi}_\theta}$, and (iii) $4\alpha(1-\alpha)\bar{\pi}_\theta\pi_\zeta \leq \frac{\lambda^2}{(H-1)^2} < \min\left\{\frac{\Pi_\theta\pi_\zeta}{\underline{\pi}_\theta}, \bar{\pi}_\theta\pi_\zeta\right\}$ and $\frac{\underline{\pi}_\theta}{\underline{\pi}_\theta+\bar{\pi}_\theta} < \alpha < \frac{1}{2}$. For example, in case (i), we can pick $D = \frac{\lambda + \sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2\bar{\pi}_\theta\pi_\zeta}}{2(1-\alpha)(H-1)\pi_\zeta}$ and we can verify $\sqrt{\frac{\underline{\pi}_\theta}{\pi_\zeta}} \leq D \leq \sqrt{\frac{\bar{\pi}_\theta}{\pi_\zeta}}$ hold. As shown before,

$$\begin{aligned}
\frac{\Pi_\theta\pi_\zeta}{\underline{\pi}_\theta} \leq \frac{\lambda^2}{(H-1)^2}. \\
\iff -\sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2\bar{\pi}_\theta\pi_\zeta} \leq 2(1-\alpha)(H-1)\sqrt{\underline{\pi}_\theta\pi_\zeta} - \lambda \\
\leq \sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2\bar{\pi}_\theta\pi_\zeta} \\
\iff \sqrt{\frac{\underline{\pi}_\theta}{\pi_\zeta}} \leq D
\end{aligned}$$

Moreover, $\frac{\lambda^2}{(H-1)^2} \leq \bar{\pi}_\theta\pi_\zeta$ implies that either

$$\sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2\bar{\pi}_\theta\pi_\zeta} \leq 2(1-\alpha)(H-1)\sqrt{\bar{\pi}_\theta\pi_\zeta} - \lambda$$

or

$$-\sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2\bar{\pi}_\theta\pi_\zeta} \geq 2(1-\alpha)(H-1)\sqrt{\bar{\pi}_\theta\pi_\zeta} - \lambda.$$

However, the latter is impossible because it implies that

$$\begin{aligned} 2(1-\alpha)(H-1)\sqrt{\bar{\pi}_\theta\pi_\zeta} - \lambda &\leq -\sqrt{\lambda^2 - 4\alpha(1-\alpha)(H-1)^2\bar{\pi}_\theta\pi_\zeta} \\ &\leq 2(1-\alpha)(H-1)\sqrt{\underline{\pi}_\theta\pi_\zeta} - \lambda \\ \implies \bar{\pi}_\theta &\leq \underline{\pi}_\theta. \end{aligned}$$

And the former implies $D \leq \sqrt{\bar{\pi}_\theta/\pi_\zeta}$. And case (ii) and (iii) are proved similarly. □

Proof of Lemma 1.4.1.

Proof. Let $(\sigma_{\theta,a}, \sigma_{\zeta,a})$ and D be a separating equilibrium. Because the linearity of the objective function and shape of the information entropy constraint set. The individual problem can be reduced to choose between two end points as discussed. It is optimal to choose to learn the payoff if $\pi_{\theta,a} \geq D^2\pi_\zeta$ and to choose to learn the supply if $\pi_{\zeta,a} \leq D^2\pi_\zeta$. Hence, we have this $\pi_\theta^* = D^2\pi_\zeta$. □

Proof of Lemma 1.4.2.

Proof. The only if part follows directly the above text. To show the if part, we can simply verify that such fixed point π_θ^* is a separating equilibrium. By the analysis in the text, we see that π_θ^* is a root of $G(\pi) = 0$ implies that one real solution for the equation $\Sigma_\zeta D^2 - \lambda D + \Sigma_\theta = 0$ must be $\sqrt{\pi_\theta^*/\pi_\zeta}$ where $\Sigma_\theta = (H-1)\int_{\pi_\theta^*}^{\bar{\pi}_\theta} \pi_\theta dF(\pi_\theta)$ and $\Sigma_\zeta = (H-1)\pi_\zeta F(\pi_\theta^*)$. Moreover, it is apparent that $\mu^*(\pi_\theta) = 0$ if $\pi_\theta < \pi_\theta^*$, $\mu^*(\pi_\theta) = 1$ if $\pi_\theta \geq \pi_\theta^*$ is optimal if agents coordinate on that asset market equilibrium $\sqrt{\pi_\theta^*/\pi_\zeta}$. □

Proof of Proposition 1.4.4.

Proof. Suppose that $\bar{\pi}_\theta\pi_\zeta < \lambda^2/(H-1)^2 < \Pi_\theta\pi_\zeta/\underline{\pi}_\theta$. Then $G(\underline{\pi}_\theta) = \Pi_\theta - \frac{\lambda}{H-1}\sqrt{\frac{\underline{\pi}_\theta}{\pi_\zeta}} > 0$ and $G(\bar{\pi}_\theta) = \bar{\pi}_\theta - \frac{\lambda}{H-1}\sqrt{\frac{\bar{\pi}_\theta}{\pi_\zeta}} < 0$, Then by continuity, there exists a $\pi_\theta^* \in (\underline{\pi}_\theta, \bar{\pi}_\theta)$ such that $G(\pi_\theta^*) = 0$. Then it is a separating equilibrium by Lemma (1.4.2). Moreover, it is unique followed by monotonicity because it must lie on the decreasing segment of the G function. The proof for the case of $\Pi_\theta\pi_\zeta/\underline{\pi}_\theta < \lambda^2/(H-1)^2 < \bar{\pi}_\theta\pi_\zeta$ is similar. □

Proof of Proposition 1.4.5.

Proof. Note that $G(0) = \Pi_\theta > 0$, $G'(0) = -\infty$, and $\lim_{\pi\to\infty} G(\pi) = \infty$. The strict convexity of G ensures that there exists a unique $\mathring{\pi}_\theta = \arg\min_\pi G(\pi)$ and it must be strictly positive because of $G'(0) = -\infty$. In fact, it satisfies $G'(\mathring{\pi}_\theta) = 0$, which gives Equation (1.4.12). Then, by continuity and convexity of G, G is continuous and decreasing on $[0, \pi^*]$ and increasing on $[\mathring{\pi}_\theta, +\infty)$, hence, if $G(\pi^*) < 0$, then there must exist a point π_θ^* on $(\mathring{\pi}_\theta, +\infty)$ and another point π_θ^* on $(0, \mathring{\pi}_\theta)$ such that $G(\pi_\theta^*) = G(\pi_\theta^*) = 0$. If $G(\mathring{\pi}_\theta) = 0$, then $\mathring{\pi}_\theta$ has to be the only zero point of G since it is the only minimum. If $G(\mathring{\pi}_\theta) > 0$, then there cannot be any zero point of G. □

Proof of Proposition 1.4.6.

Proof. Let $\pi_\theta^{*,1}$ and $\pi_\theta^{*,2}$ be two separating equilibria with $\pi_\theta^{*,1} > \pi_\theta^{*,2}$. From the proof of Proposition 1.4.5, we show that there is one separating equilibrium $\pi_\theta^{*,1}$ on $[\mathring{\pi}_\theta, +\infty)$ and another separating equilibrium$\pi_\theta^{*,2}$ on $[0, \mathring{\pi}_\theta]$. And by the definition of $\mathring{\pi}_\theta = \arg\min_\pi G(\pi)$ and properties of G that we state in the proof of Proposition 1.4.5. G is strictly increasing on $[\mathring{\pi}_\theta, +\infty)$ and decreasing on $[0, \mathring{\pi}_\theta]$, therefore, $G'(\pi_\theta^{*,1}) > 0$ and $G'(\pi_\theta^{*,2}) < 0$. Then, we have

$$G'(\pi_\theta^{*,1}) = F(\pi_\theta^{*,1}) - \frac{\lambda}{2(H-1)\sqrt{\pi_\theta^{*,1}\pi_\zeta}} > 0, \tag{A.1.5}$$

$$G'(\pi_\theta^{*,2}) = F(\pi_\theta^{*,2}) - \frac{\lambda}{2(H-1)\sqrt{\pi_\theta^{*,2}\pi_\zeta}} < 0. \tag{A.1.6}$$

The separating equilibrium π_θ^* characterized by the root of $G(\pi) = 0$, i.e. π_θ^*, λ, and H jointly satisfy:

$$J(\pi_\theta^*, \lambda, H) = \int_{\pi_\theta^*}^{\bar{\pi}_\theta} \pi_\theta dF(\pi_\theta) + \pi_\theta^* F(\pi_\theta^*) - \frac{\lambda}{H-1}\sqrt{\frac{\pi_\theta^*}{\pi_\zeta}} = 0. \tag{A.1.7}$$

Apply the Implicit Function Theorem to obtain:

$$\begin{aligned} \frac{\partial \pi_\theta^*}{\partial \lambda} &= -\frac{\partial J(\pi_\theta^*, \lambda, H)/\partial \lambda}{\partial J(\pi_\theta^*, \lambda, H)/\partial \pi_\theta^*} \\ &= \frac{\frac{1}{H-1}\sqrt{\pi_\theta^*/\pi_\zeta}}{F(\pi_\theta^*) - \frac{\lambda}{2(H-1)\sqrt{\pi_\theta^*\pi_\zeta}}}. \end{aligned} \tag{A.1.8}$$

From inequality (A.1.5) and (A.1.6), it implies that $\partial\pi_\theta^{*,1}/\partial\lambda > 0$ and $\partial\pi_\theta^{*,2}/\partial\lambda < 0$. With similar argument, it is easy to see that the comparative statics of separating equilibria with respect to H is exactly opposite: $\partial\pi_\theta^{*,1}/\partial H < 0$ and $\partial\pi_\theta^{*,2}/\partial H > 0$. □

Proof of Theorem 1.5.1.

Proof. Suppose that λ_a is bounded below by $\underline{\lambda} > 0$ and K_a is bounded above by $(\bar{\sigma}_\theta, \bar{\sigma}_\zeta)$ for almost every agent. $\underline{\lambda} \geq 2\sqrt{\bar{\sigma}_\theta\bar{\sigma}_\zeta}$ implies that $\underline{\lambda}^{-2}\bar{\sigma}_\theta\bar{\sigma}_\zeta \leq 1/4$, Then$\hat{\Sigma}_\theta\hat{\Sigma}_\zeta \leq \underline{\lambda}^{-2}\bar{\sigma}_\theta\bar{\sigma}_\zeta \leq 1/2$. This guarantees that the correspondence Γ defined in Section 1.3 has the same domain and range. Then follow the proof of Theorem 2 in Rath (1992), we can apply Kakutani's fixed point theorem to show there exists some fixed point of Γ. By Lemma 1.5.1, there exists an equilibrium. □

A.2 On the Information Constraint

Linear Constraints

One type of constraint considered by Van Nieuwerburgh and Veldkamp (2010) and Kacperczyk et al. (2016) is an additive constraint.

$$\sigma_{\theta,a} + \sigma_{\zeta,a} \leq H_a. \tag{A.2.1}$$

The interpretation for this constraint is that the learning technology is analogous to a sequence of independent draws of either a payoff or a supply signal with precision δ. Each independent draw of a normally distributed signal adds precision δ to the posterior belief. Constraining the sum of incremental precisions of the posterior belief not exceeding H_a is equivalent to restrict the total number of draws on payoff and supply signals to be $N \leq H_a/\delta$.

If the constraint is linear, no matter what heterogeneity is introduced, the equilibrium will be either all-fundamentalist or all-chartist in most cases. There is a knife-edge case of mixing equilibrium only when the asset market equilibrium $D = 1$.

Quadratic Constraints

Another possible information constraint studied by Farboodi and Veldkamp (2016) is a quadratic constraint,

$$\left(\pi_{\theta,a} + \sigma_{\theta,a}\right)^2 + \chi\left(\pi_{\zeta,a} + \sigma_{\zeta,a}\right)^2 \leq H_a. \tag{A.2.2}$$

This constraint captures idea that it is getting tougher and tougher to acquire more and more precise information about a given random variable, while the total cost of acquiring two different types of information is additive.

If the constraint is quadratic, information has decreasing returns. In the case that agents are homogeneous, there will be still multiple equilibria, but it is not guaranteed that the equilibrium is at the corners of the constraint set and it is most likely individual learn both type information in the equilibrium. With heterogeneity in prior precision introduced, the equilibrium will still be monotone, but decreasing in $\pi_{\theta,a}$, because of the decreasing return of information.

The shaded area in the Figure 1.3.1 depicts some examples of the three types of information constraint sets discussed above. Then main distinctions among them is the convexity of the constraint sets. The entropy-type constraint has a convex complement, while the quadratic constraint set itself is convex. Though the solutions to constraint optimization problems also depends on the objective function, the entropy-type constraint more likely admits corner solutions. And this is exactly the case when later we solve the attention allocation problem where the objective function is effectively linear.

Appendix B

Appendix to Chapter 2

B.1 Proofs in Chapter 2

Proof of Proposition 2.3.1.

Proof. The value function M to Problem (2.3.1) satisfies the following Bellman equation:

$$\lambda M = \frac{1}{2}\sigma^2 y M',$$

which has a general solution of the form

$$M(y) = A_1 y^\beta + A_2 y^{\beta'}$$

where β and β' are two roots of the quadratic equation $\lambda = \frac{1}{2}\sigma^2\beta(\beta-1)$, i.e.

$$\beta = \frac{1+\sqrt{1+8\lambda/\sigma^2}}{2} > 1,$$
$$\beta' = \frac{1-\sqrt{1+8\lambda/\sigma^2}}{2} < 0.$$

Note that we have a boundary condition for the value function $M(0) = 0$ so that it has to be that $A_2 = 0$. Moreover, the standard "*value matching*" (B.1.1) and "*smooth pasting*" (B.1.2) conditions are also necessary:

$$M(Y_M) = \frac{Y_M \pi_L}{\lambda} - c; \tag{B.1.1}$$
$$M'(Y_M) = \frac{\pi_L}{\lambda}. \tag{B.1.2}$$

Therefore, A_1 and Y_M jointly solve

$$A_1 Y_M^\beta = \frac{Y_M \pi_L}{\lambda} - c,$$
$$\beta A_1 Y_M^{\beta-1} = \frac{\pi_L}{\lambda},$$

which yields

$$Y_M = \frac{\beta}{\beta-1}\frac{\lambda c}{\pi_L},$$
$$A_1 = \frac{Y_M^{1-\beta}}{\beta}\frac{\pi_L}{\lambda}.$$

Then, the problem is solved and the solution is as stated in the proposition. $\square$

Proof of Proposition 2.3.2.

Proof. Suppose that $\hat{\theta}_2 > 0$ or $q < 1$, then

$$\pi_D^s - \pi_{F,s} = p(1-p)\left[\theta_2 - (2q-1)\theta_0\right]/\rho^s > 0$$

for both $s = 1$ and 2. Applying similar solution method in the proof of Proposition 2.3.2, we obtain the value function

$$F^s(y) = \begin{cases} \frac{Y_F^s\left(\pi_D^s - \pi_F^s\right)}{\beta\lambda}\left(\frac{y}{Y_F^s}\right)^\beta + \frac{y\pi_F^s}{\lambda} & y < Y_F^s, \\ \frac{y\pi_D^s}{\lambda} - c & y \geq Y_F^s, \end{cases}$$

where

$$Y_F^s = \frac{\beta}{\beta-1}\frac{\lambda c}{\pi_D^s - \pi_F^s},$$

and the timing decision

$$\tau_F^s = \inf\left\{t \geq 0 | Y_t \geq Y_F^s\right\}.$$

Moreover, if $\hat{\theta}_2 = 0$ and $q \to 1$ or $\hat{\theta}_2 \to 0$ and $q = 1$, then $(\pi_D^s - \pi_{F,s}) \to 0$ and consequently $Y_{F,s} \to \infty$ and $\tau_{F,s} \to \infty$. $\square$

Proof of Lemma 2.3.1.

Proof. The second part obviously follows the equation (2.2.11). Let's focus on the first part. We will show the case that $Y_F^1 > Y_F^0$, and case where $Y_F^1 < Y_F^0$ is analogous.

First, I claim that

$$F(y;q) = \max_{\tilde{Y}_F^0, \tilde{Y}_F^1 \geq y}\left\{\rho^1\left\{\frac{y\pi_F^1}{\lambda} + \left(\frac{y}{\tilde{Y}_F^1}\right)^\beta\left[\frac{\tilde{Y}_F^1\left(\pi_D^1 - \pi_F^1\right)}{\lambda} - c\right]\right\}\right.$$
$$\left. + \rho^0\left\{\frac{y\pi_{F,1}}{\lambda} + \left(\frac{y}{\tilde{Y}_F^0}\right)^\beta\left[\frac{\tilde{Y}_F^0\left(\pi_D^1 - \pi_F^0\right)}{\lambda} - c\right]\right\}\right\}.$$

To see this, we exploit the equivalence of an optimal stopping problem to a problem choose threshold directly.

Then, we can apply the envelop theorem and plug the expression of π's and ρ:

$$\begin{aligned}
\frac{\partial}{\partial q}F(y;q) &= \frac{\partial}{\partial q}\Bigg\{\frac{y}{\lambda}(2q-1)p(1-p)\theta_0 + \left(\frac{y}{Y_F^1}\right)^{\beta}\left[\frac{Y_F^1 p(1-p)\left[\theta_2-(2q-1)\right]}{\lambda} - \left[pq+(1-p)(1-q)\right]\right. \\
&\quad + \frac{y}{\lambda}(2q-1)p(1-p)\theta_0 + \left(\frac{y}{Y_F^0}\right)^{\beta}\left[\frac{Y_F^0 p(1-p)\left[\theta_2-(2q-1)\right]}{\lambda} - \left[p(1-q)+(1-p)q\right]c\right]\Bigg\} \\
&= \frac{4y}{\lambda}p(1-p)\theta_0 + \left(\frac{y}{Y_F^1}\right)^{\beta}\left[-\frac{Y_F^1 2p(1-p)}{\lambda} - (2p-1)c\right] + \left(\frac{y}{Y_F^0}\right)^{\beta}\left[-\frac{Y_F^0 2p(1-p)}{\lambda} - (1-\right. \\
&> \frac{4y}{\lambda}p(1-p)\theta_0 - \left[\left(\frac{y}{Y_F^0}\right)^{\beta}\frac{Y_F^1 2p(1-p)}{\lambda} + \left(\frac{y}{Y_F^0}\right)^{\beta}\frac{Y_F^0 2p(1-p)}{\lambda}\right] \\
&= \frac{4}{\lambda}p(1-p)\theta_0\left[y - \frac{\left(y/Y_F^1\right)^{\beta}Y_F^1 + \left(y/Y_F^0\right)^{\beta}Y_F^0}{2}\right] \geq 0,
\end{aligned}$$

where the first inequality follows from $Y_F^1 > Y_F^0$, which also implies that $2p-1 \geq 0$, and the second inequality is due to that $y \leq Y_F^s$ and $\beta > 1$ implies that $\left(y/Y_F^s\right)^{\beta} Y_F^s \leq y$. □

Proof of Lemma 2.3.2.

Proof. I will show the first statement, and the argument for the other statements is proved in the similar way and hence omitted.

Suppose now $1/2 < p < 1$. First, we note that $\rho^1 = pq + (1-p)(1-q) = (1-q) + (2q-1)p$ is increasing in p given that $q \geq 1/2$ and $\rho = 1/2$ if $p = 1/2$. Hence, we must have $\rho \geq 1/2$. Therefore, $\rho^1 \geq \rho^0$, and

$$\begin{aligned}
\pi_D^1 - \pi_F^1 &= \frac{p(1-p)\left[\theta_2-(2q-1)\theta_0\right]}{\rho^1} \\
&\leq \frac{p(1-p)\left[\theta_2-(2q-1)\theta_0\right]}{\rho^0} = \pi_D^0 - \pi_F^0.
\end{aligned}$$

Consequently,

$$\begin{aligned}
Y_F^1 &= \frac{\beta}{\beta-1}\frac{\lambda c}{\pi_D^1-\pi_F^1} \\
&\geq \frac{\beta}{\beta-1}\frac{\lambda c}{\pi_D^0-\pi_F^0} = Y_F^0.
\end{aligned}$$

Then, we note that

$$\begin{aligned}
\frac{\partial}{\partial q}\left(\pi_D^1-\pi_F^1\right) &= \frac{-2p(1-p)\theta_0\rho - p(1-p)\left[\theta_2-(2q-1)\theta_0\right](2p-1)}{(\rho^1)^2} < 0, \\
\frac{\partial}{\partial q}\left(\pi_D^0-\pi_F^0\right) &= \frac{p(1-p)\left[(2p-1)\theta_2-\theta_0\right]}{(\rho^0)^2} > 0.
\end{aligned}$$

Finally, to see the comparative statics,

$$
\begin{aligned}
\frac{\partial Y_F^1}{\partial q} &= -\frac{\beta\lambda c}{\beta-1}\frac{\partial\left(\pi_D^1-\pi_F^1\right)/\partial q}{(\pi_D^1-\pi_{F,1})^2} > 0,\\
\frac{\partial Y_F^0}{\partial q} &= -\frac{\beta\lambda c}{\beta-1}\frac{\partial\left(\pi_D^0-\pi_F^0\right)/\partial q}{(\pi_D^0-\pi_F^0)^2} < 0.
\end{aligned}
$$

□

Proof of Proposition 2.3.3

Proof. First, V satisfies the differential equation $\lambda V = \frac{1}{2}\sigma^2 y^2 V''$, and, along with the boundary condition $V(0) = 0$, we know it has a general solution $V(y) = A_L y^\beta$. So we solve Y_L and A_L by the the value matching and smooth pasting conditions:

$$
V(Y_L) = \begin{cases}
\frac{Y_L\pi_L}{\lambda} + \left(\frac{Y_L}{\hat{Y}_F}\right)^\beta \frac{\hat{Y}_F(\pi_D-\pi_L)}{\lambda} - c & Y_L < \min\{Y_F^0, Y_F^1\}\\
\frac{Y_L(\pi_D+\pi_L)}{2\lambda} + \left(\frac{Y_L}{Y_F^0}\right)^\beta \frac{Y_F^0(\pi_D-\pi_L)}{2\lambda} - c & Y_F^1 \le Y_L < Y_F^0\\
\frac{Y_L(\pi_D+\pi_L)}{2\lambda} + \left(\frac{Y_L}{Y_F^1}\right)^\beta \frac{Y_F^1(\pi_D-\pi_L)}{2\lambda} - c & Y_F^0 \le Y_L < Y_F^1
\end{cases}
$$

and

$$
V'(Y_L) = \begin{cases}
\frac{\pi_L}{\lambda} + \beta\left(\frac{Y_L}{\hat{Y}_F}\right)^{\beta-1} \frac{(\pi_D-\pi_L)}{\lambda} & Y_L < \min\{Y_F^0, Y_F^1\}\\
\frac{(\pi_D+\pi_L)}{2\lambda} + \beta\left(\frac{Y_L}{Y_F^0}\right)^{\beta-1} \frac{(\pi_D-\pi_L)}{2\lambda} & Y_F^1 \le Y_L < Y_F^0\\
\frac{(\pi_D+\pi_L)}{2\lambda} + \beta\left(\frac{Y_L}{Y_F^1}\right)^{\beta-1} \frac{(\pi_D-\pi_L)}{2\lambda} & Y_F^0 \le Y_L < Y_F^1.
\end{cases}.
$$

Then, we discuss different cases: Suppose that $Y_L < \min\{Y_F^0, Y_F^1\}$, then we solve $Y_L = \frac{\beta}{\beta-1}\frac{\lambda c}{\pi_L} = Y_M$ by the value matching and smooth pasting conditions; and to have $\frac{\beta}{\beta-1}\frac{\lambda c}{\pi_L} < \min\{Y_F^0, Y_F^1\}$, it is necessary and sufficient to have $\pi_L > \max_s\{\pi_D^s - \pi_F^s\}$. So we can verify that, if $\pi_L > \max_s\{\pi_D^s - \pi_F^s\}$, $Y_L = \frac{\beta}{\beta-1}\frac{\lambda c}{\pi_L}$ does satisfy the value matching and smooth pasting conditions.

Similarly, suppose that $Y_F^1 \le Y_L < Y_F^0$, then we solve $Y_L = \frac{\beta}{\beta-1}\frac{2\lambda c}{\pi_L+\pi_D}$ by the value matching and smooth pasting conditions. Now, if $\pi_L \le \pi_D^1 - \pi_F^1$, then $\frac{\beta}{\beta-1}\frac{\lambda c}{\pi_M} \ge Y_F^1$; moreover, $\pi_D \le \pi_L \implies \frac{\pi_L+\pi_D}{2} \le \pi_L$, and thus $Y_L = \frac{\beta}{\beta-1}\frac{2\lambda c}{\pi_L+\pi_D} \ge \frac{\beta}{\beta-1}\frac{\lambda c}{\pi_L} \ge Y_{F,1}$. Moreover, it must be the case that $\frac{\beta}{\beta-1}\frac{2\lambda c}{\pi_L+\pi_D} < Y_{F,0}$ because we can show that $\frac{\pi_L+\pi_D}{2} > \pi_D^0 - \pi_F^0$: First,

$$
\begin{aligned}
&\pi_L \ge \pi_D \text{ and } \pi_F \ge 0\\
\implies &\pi_L \ge \pi_D - 2\pi_F\\
\implies &\frac{\pi_L+\pi_D}{2} \ge \pi_D - \pi_F;
\end{aligned}
$$

moreover, note that $\pi_D - \pi_F = p^1(\pi_D^1 - \pi_F^1) + p^0(\pi_D^0 - \pi_F^0)$ and $Y_F^0 < Y_F^1$ implies that $\pi_D^1 - \pi_F^1 > \pi_D^0 - \pi_F^0$, hence, $\pi_D^1 - \pi_F^1 > \pi_D - \pi_F > \pi_D^0 - \pi_F^0$. Therefore, $\frac{\pi_L + \pi_D}{2} > \pi_D^0 - \pi_F^0$. And similar analysis applies to the case that $Y_F^0 \leq Y_L < Y_F^1$. □

Proof of Lemma 2.4.1.

Proof. First, note that $q = 1/2$, $F(y; \frac{1}{2}) = S(y)$ for all $y \geq 0$. Therefore, the conclusion follows Lemma 2.3.1. □

Proof of Lemma 2.4.2.

Proof. First, we note that $\pi_F = \pi_F^s = 0$ and $\pi_D = \pi_D^s$ for $s = 0, 1$. Moreover $Y_F^0 = Y_F^1 = \frac{\beta}{\beta - 1}\frac{\lambda c}{\pi_D} \triangleq Y_F$. And it is for sure that $Y_M = \frac{\beta}{\beta-1}\frac{\lambda c}{\pi_L} < Y_F$ since $\hat{\theta}_1 > \hat{\theta}_2 \geq 0$. Hence, we have $Y_L = Y_M$,

$$L(Y_L) = \frac{Y_L \pi_L}{\lambda} + \left(\frac{Y_L}{Y_F}\right)^\beta \frac{Y_F(\pi_D - \pi_L)}{\lambda} - c.$$

And

$$F(Y_L) = \left(\frac{Y_L}{Y_F}\right)^\beta \left[\frac{Y_F \pi_D}{\lambda} - c\right].$$

Therefore,

$$\begin{aligned} L(Y_L) - F(Y_L) &= \frac{Y_L \pi_L}{\lambda} + \left(\frac{Y_L}{Y_F}\right)^\beta \left(c - \frac{Y_F \pi_L}{\lambda}\right) - c \\ &= \frac{c}{\beta - 1}\left[\left(\frac{\pi_D}{\pi_L}\right)^\beta \left(\beta - 1 - \beta\frac{\pi_L}{\pi_D}\right) + 1\right] \\ &> 0 \end{aligned}$$

where, for last inequality, we show that $f(x) = (\beta - 1)x^\beta - \beta x^{\beta - 1} + 1$ is decreasing in x when $0 < x < 1$ and $f(1) = 0$. Note that

$$f'(x) = \beta(\beta - 1)x^{\beta - 2}(x - 1) < 0$$

when $0 < x < 1$. □

Proof of Lemma 2.4.3.

Proof. First, we note that if $q = 1$, then $\pi_D^1 = \pi_{D,1}$ and $\pi_D^0 = \pi_{D,0}$. Therefore,

$$L_s^s(y) - F^s(y) - c = \begin{cases} \frac{y\left(\pi_{L,s} - \pi_F^s\right)}{\lambda} + \left(\frac{y}{Y_F^s}\right)^\beta \left[c - \frac{Y_F^s\left(\pi_{L,s} - \pi_F^s\right)}{\lambda}\right] - c & y < Y_F^s \\ 0 & y \geq Y_F^s \end{cases}.$$

We consider the case where $\pi_L > \max_s \{\pi_D^s - \pi_F^s\}$. Note that in this case

$$\begin{aligned}\pi_L > \pi_D^1 - \pi_F^1 &\iff 2p\theta_1 > \hat{\theta}_2 \\ \pi_L > \pi_D^0 - \pi_F^0 &\iff 2(1-p)\theta_1 > \hat{\theta}_2.\end{aligned}$$

And $Y_L = Y_M = \frac{\beta}{\beta-1}\frac{\lambda c}{\pi_L}$. Then,

$$\begin{aligned}L_s^s(Y_L) - F^s(Y_L) - c &= c\left[\frac{\beta}{\beta-1}\frac{\pi_{L,s} - \pi_F^s}{\pi_L} + \left(\frac{\pi_D^s - \pi_F^s}{\pi_L}\right)^\beta \left(1 - \frac{\beta}{\beta-1}\frac{\pi_{L,s} - \pi_F^s}{\pi_D^s - \pi_F^s}\right) - 1\right] \\ &= \begin{cases} c\left[\frac{\beta}{\beta-1}\frac{\hat{\theta}_1}{2p\theta_1} + \left(\frac{\hat{\theta}_2}{2p\theta_1}\right)^\beta \left(1 - \frac{\beta}{\beta-1}\frac{\hat{\theta}_1}{\hat{\theta}_2}\right) - 1\right] & s = 1 \\ c\left[\frac{\beta}{\beta-1}\frac{\hat{\theta}_1}{2(1-p)\theta_1} + \left(\frac{\hat{\theta}_2}{2(1-p)\theta_1}\right)^\beta \left(1 - \frac{\beta}{\beta-1}\frac{\hat{\theta}_1}{\hat{\theta}_2}\right) - 1\right] & s = 0 \end{cases}\end{aligned}$$

Hence,

$$\begin{aligned}L(Y_L) - F(Y_L) &= p\left[L_{1,1}(Y_L) - F_1(Y_L) - c\right] + (1-p)\left[L_{2,2}(Y_L) - F_2(Y_L) - c\right] \\ &= c\left[h\left(\hat{\theta}_1/\theta_1, \hat{\theta}_2/\theta_1\right) - 1\right].\end{aligned}$$

Therefore, $h\left(\hat{\theta}_1/\theta_1, \hat{\theta}_2/\theta_1\right) < 1$ is necessary and sufficient for $L(Y_L) - F(Y_L) < 0$ in this case.

Now we consider the case where $\pi_L \le \pi_D^1 - \pi_F^1$. In this case $Y_L = \frac{\beta}{\beta-1}\frac{2\lambda c}{\pi_L + \pi_D} \in [Y_F^1, Y_F^0]$. Then, $L_1^1(Y_L) - F^1(Y_L) = 0$ and

$$\begin{aligned}L_0^0(Y_L) - F^0(Y_L) &= c\left[\frac{\beta}{\beta-1}\frac{\pi_{L,0} - \pi_F^0}{(\pi_L + \pi_D)/2} + \left(\frac{\pi_D^0 - \pi_F^0}{(\pi_L + \pi_D)/2}\right)^\beta \left(1 - \frac{\beta}{\beta-1}\frac{\pi_{L,0} - \pi_F^0}{\pi_D^0 - \pi_F^0}\right) - 1\right] \\ &= c\left[\frac{\beta}{\beta-1}\frac{\hat{\theta}_1}{(1-p)(\theta_1+\theta_2)/2} + \left(\frac{\hat{\theta}_2}{(1-p)(\theta_1+\theta_2)/2}\right)^\beta \left(1 - \frac{\beta}{\beta-1}\frac{\hat{\theta}_1}{\hat{\theta}_2}\right) - 1\right] \\ &= c\left[g_0\left(\frac{\hat{\theta}_1}{(\theta_1+\theta_2)/2}, \frac{\hat{\theta}_2}{(\theta_1+\theta_2)/2}\right) - 1\right].\end{aligned}$$

Therefore, $g_0\left(\frac{\hat{\theta}_1}{(\theta_1+\theta_2)/2}, \frac{\hat{\theta}_2}{(\theta_1+\theta_2)/2}\right) < 1$ is necessary and sufficient for $L(Y_L) - F(Y_L) < 0$. The case where $\pi_L \le \pi_D^0 - \pi_F^0$ is similar, so we omit it. □

Proof of Lemma 2.4.4.

Proof. This follows the fact that $\phi(y) = L(y) - F(y)$ is strictly concave on $[0, \max\{Y_F^0, Y_F^1\}]$, which has at most two zero points, $\phi(0) < 0$, $\phi(Y_L) > 0$, $\phi(\max\{Y_F^0, Y_F^1\}) = 0$.

First, we can verify that L' is strictly decreasing in y and F' is strictly increasing in y on $[0, \max\{Y_F^0, Y_F^1\}]$, therefore, L and $-F$ are strictly concave, so that ϕ is concave on this region. Moreover, $\phi(0) = -c < 0$. $\phi(Y_L) > 0$ by assumption. Hence there must exists some $Y_P \in (0, Y_L)$ such that $\phi(Y_P) = 0$. Moreover, since we know that $\phi\left(\max\left\{Y_F^0, Y_F^1\right\}\right) = 0$, this Y_P has to be unique, otherwise it contradicts with the concavity of ϕ. □

Proof of Proposition 2.4.3.

Proof. First, we show that the strategy profile (τ_1^*, τ_2^*) is an equilibrium by showing that take $\tau_{-i}^* = \inf\{t \geq 0 : Y_t \geq Y_P\}$ as given, $\tau_i^* = \inf\{t \geq 0 : Y_t \geq Y_P\}$ is a best response. Consider agent i is choosing some threshold Y_i.

If $y \geq Y_P$, choosing the threshold $Y_i \leq y$ will not change the outcome. Moreover, i has no incentive to chooses the threshold $Y_i > y$ either since

$$U_i(\tau_i^*, \tau_{-i}^*, y) = \left[L(y) + F(y)\right]/2 \geq F(y) = U_i(\tau_i^*, \tau_{-i}^*, y),$$

for all $y > Y_P$.

Suppose that $y < Y_P$. Then choosing the threshold $Y_i > Y_P$ will not change the outcome. Moreover, i has no incentive to choose the threshold $Y_i < Y_P$ because in that case $U_i(\tau_i, \tau_{-i}^*, y) = E\left[e^{-\lambda\tau_i}L(Y_{\tau_i})|Y_0 = y\right] < E\left[e^{-\lambda\tau_j^*}L(Y_{\tau_j^*})|Y_0 = y\right] = U_j(\tau_i^*, \tau_j^*, y)$ by the fact that $E\left[e^{-\lambda\tau_i}L(Y_{\tau_i})|Y_0 = y\right]$ is strictly increasing in Y_i when $Y_i \in [0, Y_L]$. Hence, we can conclude that τ_i^* is a best response.

Second, we rule out the possibility of other equilibrium of this class. Suppose we have another equilibrium such that $\tau_i = \inf\{t : Y_t \geq Y_i\}$ and $\tau_{-i} = \inf\{t : Y_t \geq Y_{-i}\}$ with either Y_i or Y_j does not equal to Y_P or neither of them equal to Y_P, and we will find possible deviations for all the following cases.

Case 1: Suppose $Y_i \leq Y_{-i} < Y_P$. When $y = Y_{-i}$, by choosing some $Y_i' > Y_{-i}$, the trader i can increase her expected payoff since $F(Y_{-i}) > L(Y_{-i})$.

Case 2: Suppose $Y_i < Y_P \leq Y_{-i}$. When $y = Y_i$, by choosing $Y_i' = Y_P$, the trader i will have expected payoff

$$U_i(\tau_i', \tau_{-i}, y) = \left(\frac{y}{Y_P}\right)^\beta L(Y_P) > L(Y_i) = U_i(\tau_i, \tau_{-i}, y)$$

since $E\left[e^{-\lambda\tau_i}L(Y_{\tau_i})|Y_0 = y\right] = \left(\frac{y}{Y_i}\right)^\beta L(Y_i)$ is strictly increasing when $Y_i < Y_L$.

Case 3: Suppose $Y_P = Y_i < Y_{-i}$. When $y = Y_i$, i will choosing some $Y_i' \in (Y_i, \min\{Y_L, Y_{-i}\})$, so her expected payoff will increase for the same reason as case 2.

Case 4: Suppose $Y_P < Y_i \leq Y_{-i}$. When $y = Y_i - \epsilon$, for some $\epsilon > 0$ small enough, choosing some $Y_{-i}' = Y_i - \epsilon$, change of the trader $-i$'s expected payoff is

$$\Delta U_{-i}(\epsilon) = L(Y_i - \epsilon) - \left(\frac{Y_i - \epsilon}{Y_i}\right)^\beta F(Y_i)$$

note that since $\lim_{\epsilon\to 0} \Delta U_{-i}(\epsilon) > 0$ and it is continuous, so when $\epsilon > 0$ small enough, we will have $\Delta U_{-i}(\epsilon) > 0$. □

Proof of Proposition 2.4.4.

Proof. We will show (i) is an equilibrium, and the proof for (ii) being an equilibrium is similar and hence omitted.

Given that $\tau_1^* = \inf\{t \geq 0 : Y_t^y \geq Y_L\}$. When $y > Y_L$, then trader 2 is indifferent because he will get follower's value any way. When $y = Y_L$, the only non-trivial deviation is to choose $\tau_2(Y_L) = 0$, however, in this case, the change of trader 2's expected payoff will be $[L(Y_L) - F(Y_L)]/2 \leq 0$ since $F(Y_L) \geq L(Y_L)$, so she does not strictly prefer to this deviation. When $y < Y_L$, i.e. trader 1 has not acquired information, trader 2 is indifferent for choosing the threshold $Y_2 > Y_L$ since he will always get the follower's value when trader 1 acquires information. And trader 2 does not strictly prefer to acquire the information at the same time of trader 1. To see this, suppose trader 2 choose $\tau_2 = \inf\{t \geq: Y_t^y \geq Y_L\}$, then

$$\begin{aligned} U_2(\tau_1^*, \tau_2, y) &= E\left[e^{-\lambda\tau_2}\frac{[L(Y_L) + F(Y_L)]}{2}|Y_0 = y\right] \\ &\leq E\left[e^{-\lambda\tau_2}F(Y_L)|Y_0 = y\right] \\ &= U_2(\tau_1^*, \tau_2^*, y) \end{aligned}$$

since $L(Y_L) \leq F(Y_L)$. Moreover, trader 2 does not prefer to lead. To see this, suppose trader 2 choose $\tau_2 = \inf\{t : Y_t \geq Y_2\}$ for some $Y_2 < Y_L$, then

$$\begin{aligned} U_2(\tau_1^*, \tau_2, y) &= E\left[e^{-\lambda\tau_2}L(Y_2)|Y_0 = y\right] \\ &= \begin{cases} L(y) & \text{if } y \geq Y_2 \\ (y/Y_2)^\beta L(Y_2) & \text{if } y < Y_2 \end{cases}, \end{aligned}$$

and since $(y/Y)^\beta L(Y)$ is strictly increasing in Y when $Y \leq Y_L$, it must be the case that $U_2(\tau_1^*, \tau_2, y) < (y/Y_L)^\beta L(Y_L)$, hence

$$U_2(\tau_1^*, \tau_2, y) < (y/Y_L)^\beta L(Y_L) \leq (y/Y_L)^\beta F(Y_L) = U_2(\tau_1^*, \tau_2^*, y).$$

Now given that $\tau_2^* = \inf\{t \geq 0 : Y_t \geq Y_*\}$. When $y \geq Y_*$, if trader 1 choose to deviate to $\tau_1 = \inf\{t \geq 0 : Y_t^y \geq Y_1\}$ for some $Y_1 > Y_*$. Then, her expected payoff is $F(Y_*)$ is equal to that of τ_1^*, which is $[L(Y_*) + F(Y_*)]/2$ since $F(Y_*) = L(Y_*)$. When $y < Y_*$, the optimal solution for trader 1's stopping problem is to choose τ_1^*, hence she has no incentive to deviate. □

Appendix C

Appendix to Chapter 3

C.1 Additional Proofs in Chapter 3

Proof of Theorem 3.2.1.

Notice that for each round of trading, the pricing strategies and trading strategies are directly responding to the public beliefs $\Pi_t(h_t)$, and the inter-period linkage of the game is just through the evolution of beliefs. Hence, first we consider each period of the game as a *stage game*. Or equivalently, given some $\Pi \subseteq \Delta(V)$, let us consider a stage game to be a one-shot trading game, that is, a trading game with $T = 1$. So we first prove the following lemma.

Lemma C.1.1. *There exists an equilibrium in a stage game for any beliefs* $\Pi \subseteq \Delta(V)$.

Proof. First note that a and b will be chosen only from $[\underline{v}, \bar{v}]^2$ where $\underline{v} = \min V$ and $\bar{v} = \max V$, implied by Proposition 3.2.1.

Then, consider the informed agent's problem, given Π,

$$\max_{\sigma=(\zeta,\iota)\in\Delta} \min_{p\in\mathcal{P},\pi\in\Pi} E^{p\times\pi}\left[\zeta u(v-a) + \iota u(b-v)|s\right]$$

satisfies the hypothesis of the Maximum Theorem (Berge, 1959), moreover,

$$\min_{p\in\mathcal{P},\pi\in\Pi} E^{p\times\pi}\left[\zeta u(v-a) + \iota u(b-v)|s\right]$$

is concave and Δ is convex, therefore the solution correspondence, $\sigma : [\underline{v}, \bar{v}]^2 \times S \rightrightarrows \Delta$ is non-empty-valued, convex-valued, and upper hemicontinuous.

Next, we define the following correspondence $\Gamma : [\underline{v}, \bar{v}]^2 \rightrightarrows [\underline{v}, \bar{v}]^2$ by that, for any $(a, b) \in [\underline{v}, \bar{v}]^2$,

$$\begin{aligned}\Gamma(a,b) = \Big\{ & (\hat{a}, \hat{b}) \in [\underline{v}, \bar{v}]^2 \, | \exists \hat{\sigma}(a,b,\cdot) : S \to \Delta \text{ s.t.} \\ & \hat{\sigma}(a,b,s) \in \sigma(a,b,s), \, \forall s \in S; \text{ and} \\ & \hat{a} = \max_{p \in \mathcal{P}, \pi \in \Pi} E^{p \times \pi} \left[v | z = 1; \hat{\zeta}(a,b,\cdot) \right] \\ & \hat{b} = \min_{p \in \mathcal{P}, \pi \in \Pi} E^{p \times \pi} \left[v | z = -1; \hat{\iota}(a,b,\cdot) \right] \Big\}.\end{aligned}$$

Then, we verify that Γ is non-empty-valued, convex-valued, and upper hemicontinuous.

Non-empty-valued. For any $(a, b) \in [\underline{v}, \bar{v}]^2$, $\sigma(a, b, s)$ is non-empty-valued for all $s \in S$, therefore, we can always pick some $\hat{\sigma}(a, b, s) \in \sigma(a, b, s)$ for all $s \in S$. Given this $\hat{\sigma}$, $E^{p\times\pi}\left[v|z = 1; \hat{\zeta}(a, b, \cdot)\right]$ and $E^{p\times\pi}[v|z = -1; \hat{\iota}(a, b, \cdot)]$ continuous and both $\mathcal{P}$ and Π are compact, there must exist some $(\hat{a}, \hat{b})$.

Convex-valued. For any $(\hat{a}, \hat{b}), (\tilde{a}, \tilde{b}) \in \Gamma(a, b)$, we want to show that $(a_\lambda, b_\lambda) \in \Gamma(a, b)$ for any $\lambda \in (0, 1)$ where $(a_\lambda, b_\lambda) = \lambda(\hat{a}, \hat{b}) + (1 - \lambda)(\tilde{a}, \tilde{b})$. That is, we want to find some $\sigma_\lambda(a, b, \cdot)$ satisfy the conditions. So the σ_λ can be constructed as follow: let $\hat{\sigma}$ and $\tilde{\sigma}$ be one corresponding trading strategy to $(\hat{a}, \hat{b})$ and $(\tilde{a}, \tilde{b})$, respectively. Consider the following function of $\lambda_a \in [0, 1]$,

$$f(\lambda_a) = \min_{p \in \mathcal{P}, \pi \in \Pi} E^{p \times \pi} \left[a_\lambda - v | z = 1; \zeta_{\lambda_a}(a,b,\cdot) \right],$$

note that $a_\lambda \in (\hat{a}, \tilde{a})$ (wlog assuming $\hat{a} < \tilde{a}$), and since $\min_{p\in\mathcal{P},\pi\in\Pi} E^{p\times\pi}[a_\lambda - v|z = 1; \zeta_{\lambda_a}(a, b, \cdot)]$ is strictly increasing in a, therefore,

$$\begin{aligned}f(0) &= \min_{p \in \mathcal{P}, \pi \in \Pi} E^{p \times \pi} \left[a_\lambda - v | z = 1; \tilde{\zeta}(a,b,\cdot) \right] \\ &< \min_{p \in \mathcal{P}, \pi \in \Pi} E^{p \times \pi} \left[\tilde{a} - v | z = 1; \tilde{\zeta}(a,b,\cdot) \right] = 0\end{aligned}$$

and $f(1) > 0$ analogously, so there exists $\lambda_a \in (0, 1)$ such that $f(\lambda_a) = 0$ by continuity. Similarly, there exist unique λ_b such that

$$g(\lambda_b) = \min_{p \in \mathcal{P}, \pi \in \Pi} E^{p \times \pi} \left[b_\lambda - v | z = 1; \iota_{\lambda_a}(a,b,\cdot) \right] = 0.$$

The last step is to show that $[\zeta_{\lambda_a}(a, b, s), \iota_{\lambda_b}(a, b, s)] \in \sigma(a, b, s)$ for all s, where $\zeta_{\lambda_a} = \lambda_a\hat{\zeta} + (1 - \lambda_a)\tilde{\zeta}$ and $\iota_{\lambda_b} = \lambda_b\hat{\iota} + (1 - \lambda_b)\tilde{\iota}$. Note that $\zeta(a, b, s)$ can be either 0, or 1, or $[0, 1]$ in each case $\zeta_{\lambda_a}(a, b, s) \in \zeta(a, b, s)$, similarly, $\iota_{\lambda_b}(a, b, s) \in \iota(a, b, s)$, moreover, $\zeta_{\lambda_a} + \iota_{\lambda_b} \leq \max\left\{\hat{\zeta} + \hat{\iota}, \tilde{\zeta} + \tilde{\iota}\right\} \leq 1$; so we have the conclusion as desired.

Upper hemicontinuity. Let $(a_n, b_n)_{n=1}^\infty \subseteq [\underline{v}, \bar{v}]^2$ be a sequence such that $\lim_n(a_n, b_n) = (a, b)$ and $(\hat{a}_n, \hat{b}_n)_{n=1}^\infty \subseteq [\underline{v}, \bar{v}]^2$ with $(\hat{a}_n, \hat{b}_n) \in \Gamma(a_n, b_n)$ and $\lim_n(\hat{a}_n, \hat{b}_n) = (\hat{a}, \hat{b})$, we want to show that $(\hat{a}, \hat{b}) \in \Gamma(a, b)$. Note that for each $(\hat{a}_n, \hat{b}_n)$, there is a corresponding function $\hat{\sigma}_n(a, b, \cdot)$ and since $(\hat{\sigma}_n)_{n=1}^{\infty} \subseteq \Delta^3$, which is compact, so there exists a limit $\hat{\sigma}$ of its some subsequence, moreover, since $\sigma(a, b, s)$ can be show is closed, so $\hat{\sigma}(a, b, s) \in \sigma(a, b, s)$, and by continuity, we will have $\min_{p \in \mathcal{P}, \pi \in \Pi} E^{p \times \pi}\left[a_\lambda - v | z = 1; \zeta_{\lambda_a}(a, b, \cdot)\right] = 0$ and $\min_{p \in \mathcal{P}, \pi \in \Pi} E^{p \times \pi}\left[b_\lambda - v | z = 1; \iota_{\lambda_a}(a, b, \cdot)\right] = 0$, hence, $(\hat{a}, \hat{b}) \in \Gamma(a, b)$ as desired.

With closed the domain and compact range, upper hemicontinuity is equivalent to have closed graph, hence, we can apply Katutani fixed point theorem, that is, Γ has a fixed point, i.e. there exists some $(a, b) \in [\underline{v}, \bar{v}]^2$ such that $(a, b) \in \Gamma(a, b)$. $\square$

Proof of the theorem

Proof. Given $\Pi \subseteq \Delta(V)$, let $x[\Pi] = (a[\Pi], b[\Pi])$ and $\sigma[\Pi] : [\underline{v}, \bar{v}]^2 \times S \rightrightarrows \Delta$ denote the equilibrium of a stage game with belief $\Pi \subseteq \Delta(V)$. Then the equilibrium of the trading game is constructed as follow inductively:

In period 1, $x_1 = x[\Pi_1]$ and $\sigma_1 = \sigma[\Pi_1]$, and it induces a set of conditional probabilities $Q_1(\pi) = \{\Pr(z_1|v; \pi, p) \in \Delta |\, p \in \mathcal{P}\}$ by fixing some $\pi \in \Pi_1$.

In period 2, we have $h_2 = (a_1, b_1, z_1)$, then

$$\Pi_2(h_2) = \Bigg\{ \pi' \in \Delta(V) | \pi'(v) = \frac{\pi(v) \Pr(z_1|v; \pi, p)}{\sum_v \pi(v) \Pr(z_1|v; \pi, p)}),$$
$$\text{for some } \pi \in \Pi_1 \text{ and } \Pr(z_1|v; \pi, p) \in Q_1(\pi \Bigg\},$$

then $x_2(h_2) = x[\Pi_2(h_2)]$ and $\sigma_2(h_2, a, b, s) = \sigma[\Pi_2(h_2)](a, b, s)$, and again it induces a set of conditional probabilities, $Q_2(\pi; h_2) = \{\Pr(z_2|v, h_2; \pi, p) \in \Delta |\, p \in \mathcal{P}\}$ by fixing some $\pi \in \Pi_2(h_2)$.

In general, in period $t > 1$, $h_t = (h_{t-1}, x_{t-1}(h_{t-1}), z_{t-1}(h_{t-1}))$, then

$$\Pi_t(h_t) = \Bigg\{ \pi' \in \Delta(V) | \pi'(v) = \frac{\pi(v) \Pr(z_t|v, h_t; \pi, p)}{\sum_v \pi(v) \Pr(z_t|v, h_t; \pi, p)},$$
$$\text{for some } \pi \in \Pi_{t-1}(h_{t-1}) \text{ and } \Pr(z_t|v, h_t; \pi, p) \in Q_t(\pi; h_t) \Bigg\},$$

then $x_t(h_t) = x[\Pi_t(h_t)]$ and $\sigma_t(h_t, a, b, s) = \sigma[\Pi_t(h_t)](a, b, s)$, $\forall a, b, s$, and it induces a set of conditional probabilities, $Q_t(\pi; h_t)$ similarly defined as before.

So at every history h_t, the above defined strategy profile satisfies the equilibrium conditions by verifying that in each stage game with belief $\Pi_t(h_t)$ given. Moreover, by the inductive definition of Π, it is clearly consistent. Hence, above defined strategy profile and belief system constitute an equilibrium. $\square$

Proof of Proposition 3.2.1

First, we show the following lemma.

Lemma C.1.2. *In an equilibrium (x, σ, Π), the trading strategy σ satisfy the following property: given a history $h_t \in H_t(x, \sigma, \Pi)$,*

(i) if $\zeta_t(h_t^x, s_t) > 0$ for some $s_t \in S$, then $\zeta_t(h_t^x, s_t') = 1$ for all $s_t' > s_t$;

(ii) if $\iota_t(h_t^x, s_t) > 0$ for some $s_t \in S_t$, then $\iota_t(h_t^x, s_t') = 1$ for all $s_t' < s_t$.

Proof. We will only show (i) and the proof of (ii) is similar. The proof simply relies on the Remark 3.2.1.

Note that $\zeta_t(h_t^x, s_t) > 0$, by optimality, implies that

$$\min_{\pi,p} E^{\pi\times p}\left[u\left(v - a_t(h_t)\right)|h_t, s_t\right] \geq \max\left\{0, \min_{\pi,p} E^{\pi\times p}\left[u\left(b_t(h_t) - v\right)|h_t, s_t\right]\right\}.$$

Fix arbitrary $s_t' > s_t$. Fix arbitraryπ, q, since $\Pr\left(v|h_t, s_t'; \pi, q\right)$ FOSD $\Pr\left(v|h_t, s_t; \pi, q\right)$ by Remark 3.2.1,

$$\begin{aligned}
E^{\pi\times p}\left[u\left(v - a_t(h_t)\right)|h_t, s_t'\right] &> E^{\pi\times p}\left[u\left(v - a_t(h_t)\right)|h_t, s_t\right];\\
E^{\pi\times p}\left[u\left(b_t(h_t) - v\right)|h_t, s_t'\right] &< E^{\pi\times p}\left[u\left(b_t(h_t) - v\right)|h_t, s_t\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\min_{\pi,p} E^{\pi\times p}\left[u\left(v - a_t(h_t)\right)|h_t, s_t'\right] &> \min_{\pi,p} E^{\pi\times p}\left[u\left(v - a_t(h_t)\right)|h_t, s_t\right];\\
\min_{\pi,p} E^{\pi\times p}\left[u\left(b_t(h_t) - v\right)|h_t, s_t'\right] &< \min_{\pi,p} E^{\pi\times p}\left[u\left(b_t(h_t) - v\right)|h_t, s_t\right].
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
\min_{\pi,p} E^{\pi\times p}\left[u\left(v - a_t(h_t)\right)|h_t, s_t'\right] &> \min_{\pi,p} E^{\pi\times p}\left[u\left(v - a_t(h_t)\right)|h_t, s_t\right]\\
&\geq \max\left\{0, \min_{\pi,p} E^{\pi\times p}\left[u\left(b_t(h_t) - v\right)|h_t, s_t\right]\right\}\\
&> \max\left\{0, \min_{\pi,p} E^{\pi\times p}\left[u\left(b_t(h_t) - v\right)|h_t, s_t'\right]\right\},
\end{aligned}$$

and $\zeta_t(h_t^x, s_t') = 1$ and, since s_t' is arbitrary, it holds for all $s_t' > s_t$. □

Proof for the proposition

Proof. The proof of (ii) would be similar, so we just prove (i) here. There can be two cases, one is that $\zeta < 1$ and other is that $\zeta = 1$. But by the above lemma, $\zeta < 1$ is not possible. Suppose, that for all signal s_t, $0 < \zeta_t(h_t^x, s_t) < 1$, but, by the lemma, $\zeta_t(h_t^x, s_t') = 1 > \zeta_t(h_t^x, s_t)$ for some fixed $s_t < \max S$, a contradiction.

Moreover, it is not possible for $\zeta = 1$, either. Given some equilibrium (x, σ, Π), suppose, for contradiction, there is some history $h_t \in H_t\left(x, \sigma, \Pi\right)$ such that $\zeta_t(h_t^x, s_t) = 1$ for all $s_t \in S$. Then, we can show that $\Pr\left(v|z_t = 1, h_t\right) = \Pr\left(v|z_t = -1, h_t\right) = \pi(v)$, so the market maker's pricing strategy will yield

$$a_t(h_t) = \max_{\pi \in \Pi(h_t)} E^{\pi}\left[v|h_t\right],$$
$$b_t(h_t) = \min_{\pi \in \Pi(h_t)} E^{\pi}\left[v|h_t\right].$$

Then, from informed trader's strategy, it must be the case that

$$\min_{\pi,p} E^{\pi\times p}\left[u\left(v - a_t(h_t)\right)|h_t, \underline{s}\right] \geq 0,$$

where $\underline{s} = \min S$, then, by Jensen's inequality, it implies that

$$\begin{aligned} &\min_{\pi,p} E^{\pi\times p}\left[v - a_t(h_t)|h_t, \underline{s}\right] \geq 0 \\ &\iff \min_{\pi,p} E^{\pi\times p}\left[v|h_t, \underline{s}\right] \geq \max_{\pi\in\Pi(h_t)} E^{\pi}\left[v|h_t\right]. \end{aligned}$$

However, given Assumption 2, we can show that $\Pr\left(v|h_t;\pi\right) = \pi(v)$ FOSD $\Pr\left(v|h_t, \underline{s}; \pi, p\right)$ for any π, p; hence, we have

$$\begin{aligned} &E^{\pi\times p}\left[v|h_t, \underline{s}\right] < E^{\pi}\left[v|h_t\right], \forall \pi, p \\ \implies &\min_{\pi,p} E^{\pi\times p}\left[v|h_t, \underline{s}\right] < \max_{\pi\in\Pi(h_t)} E^{\pi}\left[v|h_t\right], \end{aligned}$$

a contradiction. □

Proof of Proposition 3.2.2

Proof. Let some equilibrium (x, σ, Π) be given. Note that, by Remark 3.2.1,

$$\min_{p\in\mathcal{P},\pi\in\Pi_t(h_t)} E^{p\times\pi}\left[u\left(v - a_t^*\right)|h_t, \bar{s}\right] \leq 0 \implies \min_{p\in\mathcal{P},\pi\in\Pi_t(h_t)} E^{p\times\pi}\left[u\left(v - a_t^*\right)|h_t, s\right] < 0, \forall s < \bar{s}$$
$$\min_{p\in\mathcal{P},\pi\in\Pi_t(h_t)} E^{p\times\pi}\left[u\left(b_t^* - v\right)|h_t, \underline{s}\right] \leq 0 \implies \min_{p\in\mathcal{P},\pi\in\Pi_t(h_t)} E^{p\times\pi}\left[u\left(b_t^* - v\right)|h_t, \bar{s}\right] < 0, \forall s > \underline{s}$$

where

$$a_t^* = \max_{\pi\in\Pi(h_t)} E^{\pi}\left[v|h_t\right],$$
$$b_t^* = \min_{\pi\in\Pi(h_t)} E^{\pi}\left[v|h_t\right].$$

(If side.) If

$$\min_{p\in\mathcal{P},\pi\in\Pi_t(h_t)} E^{p\times\pi}\left[u\left(v - a_t^*\right)|h_t, \bar{s}\right] \leq 0, \text{ and}$$
$$\min_{p\in\mathcal{P},\pi\in\Pi_t(h_t)} E^{p\times\pi}\left[u\left(b_t^* - v\right)|h_t, \underline{s}\right] \leq 0,$$

holds, we can verify that

$$\begin{aligned} a_t(h_t) &= a_t^* \\ b_t(h_t) &= b_t^* \\ \zeta_t(h_t^x, s_t) = \iota_t(h_t^x, s_t) &= 0 \ \forall s_t \in S \end{aligned}$$

satisfy the equilibrium condition, hence a_t^* and b_t^* are a pair of equilibrium prices given history h_t, then $\zeta_t(h_t^x, s_t) = \iota_t(h_t^x, s_t) = 0$, $\forall s_t \in S$. Therefore, there is an informational cascade by its definition.

(Only if side.) If there is an informational cascade, then by Lemma (3.2.1) it must be the case that $\zeta_t(h_t^x, s_t) = \iota_t(h_t^x, s_t) = 0$, $\forall s_t \in S$. Then, note that for any $\pi \in \Pi(h_t)$

$$\Pr(v|z_t = 1, h_t; \pi) = \frac{\frac{1-\eta}{3}\pi(v)}{\sum_v \frac{1-\eta}{3}\pi(v)} = \pi(v)$$

and similarly, $\Pr(v|z_t = -1, h_t; \pi) = \pi(v)$, for all $v \in V$. So the equilibrium price would be, by Proposition 3.2.1,

$$\begin{aligned} a_t(h_t) &= \max_{\pi,p} E^{p\times\pi}[v|z_t = 1, h_t] \\ &= \max_{\pi} E^{\pi}[v|h_t] \\ &= a_t^*, \end{aligned}$$

and similarly $b_t(h_t) = b_t^*$. Moreover, it is easy to check that

$$\begin{aligned} & \zeta_t(h_t^x, s) = 0, \ \forall s \\ \implies & \min_{p\in\mathcal{P},\pi\in\Pi_t(h_t)} E^{p\times\pi}[u(v - a_t^*)|h_t, s] \le 0, \forall s \\ \implies & \min_{p\in\mathcal{P},\pi\in\Pi_t(h_t)} E^{p\times\pi}[u(v - a_t^*)|h_t, \bar{s}] \le 0, \end{aligned}$$

and analogously,

$$\min_{p\in\mathcal{P},\pi\in\Pi_t(h_t)} E^{p\times\pi}[u(b_t^* - v)|h_t, \underline{s}] \le 0.$$

□

Proof of Theorem 3.2.2

Proof. Given some equilibrium (x, σ, Π), let

$$\begin{aligned} a_t^* &= \max_{\pi\in\Pi(h_t)} E^{\pi}[v|h_t], \\ b_t^* &= \min_{\pi\in\Pi(h_t)} E^{\pi}[v|h_t]. \end{aligned}$$

Then,

$$\begin{aligned}
&\min_{p\in\mathcal{P},\pi\in\Pi_t(h_t)} E^{p\times\pi}\left[v|h_t,\bar{s}\right] \le a_t^* \\
\implies &\min_{p\in\mathcal{P},\pi\in\Pi_t(h_t)} E^{p\times\pi}\left[(v-a_t^*)\,|h_t,\bar{s}\right] \le 0 \\
\implies &\min_{p\in\mathcal{P},\pi\in\Pi_t(h_t)} E^{p\times\pi}\left[u\,(v-a_t^*)\,|h_t,\bar{s}\right] \le u\left[\min_{p\in\mathcal{P},\pi\in\Pi_t(h_t)} E^{p\times\pi}\left[(v-a_t^*)\,|h_t,\bar{s}\right]\right] \\
&\le u(0)=0,
\end{aligned}$$

by Jensen's inequality and Assumption 1. And sufficiency of inequality (3.2.7) is analogous. □

Proof of Proposition 3.2.3

First, we show a lemma.

Lemma C.1.3. *Given any equilibrium, it is impossible to have $\zeta_t(h_t^x,\underline{s}) > 0$ or $\iota_t(h_t^x,\bar{s}) > 0$ for any history h_t in equilibrium.*

Proof. I will show $\iota_t(h_t^x,\bar{s}) > 0$ here and the other is similar. Suppose that $\iota_t(h_t^x,\bar{s}) > 0$, for contradiction, then

$$\begin{aligned}
&\min_{p\in\mathcal{P},\pi\in\Pi_t(h_t)} E^{p\times\pi}\left[u(b_t-v)|h_t,\bar{s}\right] \ge 0 \\
\implies & b_t \ge \max_{\pi,p} E^{\pi\times p}\left[v|h_t,\bar{s}\right].
\end{aligned}$$

However, by Proposition 3.2.1,

$$\begin{aligned}
b_t &= \min_{\pi,p} E^{p\times\pi}\left[v|h_t,z_t=-1\right] \\
&< \min_{\pi,p} E^{p\times\pi}\left[v|h_t,\bar{s}\right] \\
&\le \max_{\pi,p} E^{p\times\pi}\left[v|h_t,\bar{s}\right] \le b_t,
\end{aligned}$$

where the first inequality due to that $\Pr(v|h_t,\bar{s})$ FOSD $\Pr(v|h_t,z_t=-1)$, by showing that

$$\frac{\Pr(\bar{s}|h_t,v')}{\Pr(\bar{s}|h_t,v')} > \frac{\Pr(z_t=-1|h_t,v')}{\Pr(z_t=-1|h_t,v')},$$

for all $v' > v$, and hence, a contradiction. □

Proof of the proposition

Proof. Then, by the above lemma, in equilibrium (x,σ,Π), given any $h_t\in H_t(x,\sigma,\Pi)$, note that one of the following cases must hold:

(i) $\zeta_t(h_t^x,s_t)=0$ and $\iota_t(h_t^x,s_t)=0$ for all $s_t\in S$ (informational cascade); or

(ii) There exists some $\underline{s} < s^* \le \bar{s}$ such that $\zeta_t(h_t^x,s^*) > 0$ and $\zeta_t(h_t^x,s)=0$ for all $s < s^*$; or

(iii) There exists some $\underline{s} \leq s_* < \bar{s}$ such that $\iota_t(h_t^x, s_*) > 0$ and $\iota_t(h_t^x, s) = 0$ for all $s > s_*$,

and (ii) and (iii) can hold simultaneously with $s_* \leq s^*$.

Then we prove the statement by cases.

If it is case (i), then $\Pi(h_t)$ must not be a singleton, therefore, $a_t(h_t) = \max_\pi E^\pi [v|h_t] \geq \min_\pi E^\pi [v|h_t] = b_t(h_t)$ and if $\max_\pi E^\pi [v|h_t] = \min_\pi E^\pi [v|h_t]$, it would be contradictory to Theorem 3.2.2.

If it is other cases, then, it suffice to show that $\Pr(v|h_t, z_t = 1)$ strictly FSOD $\Pr(v|h_t, z_t = -1)$ by showing that in each case

$$\frac{\Pr(z_t = 1|h_t, v')}{\Pr(z_t = 1|h_t, v)} > \frac{\Pr(z_t = -1|h_t, v')}{\Pr(z_t = -1|h_t, v)},$$

which is true because of the informativeness assumption of the signals.

Therefore, we have

$$\begin{aligned} a_t(h_t) &= \max_{\pi,p} E^{p\times\pi} [v|h_t, z_t = 1] \\ &> \max_{\pi,p} E^{p\times\pi} [v|h_t, z_t = -1] \\ &\geq \min_{\pi,p} E^{p\times\pi} [v|h_t, z_t = -1] \\ &= b_t(h_t). \end{aligned}$$

□

Proof of Corollary 3.2.1

Proof. Note that if there is an informational cascade after history h_t, first, then it must be that $\min_{\pi\in\Pi(h_t)} E^\pi [v|h_t] < \max_{\pi\in\Pi(h_t)} E^\pi [v|h_t]$; second, $\Pi_{t+1}(h_{t+1}) = \Pi_t(h_t)$ since we can show that

$$\Pr(v|h_t, z_t = 1; \pi, p) = \Pr(v|h_t, z_t = 1; \pi, p) = \pi(v).$$

And by induction $\Pi(h_\tau) = \Pi(h_t)$ for all $\tau \geq t$ and $h_\tau \in H_\tau(x, \sigma, \Pi)$. And the corollary then is an implication of Proposition 3.2.2 and 3.2.3. □

Proof of Theorem 3.2.3

Lemma C.1.4. *In any equilibrium (x, σ, Π), it is impossible to have both $\zeta_t(h_t^x, s_t)$ and $\iota_t(h_t^x, s_t)$ strictly positive for any history $h_t \in H_t(x, \sigma, \Pi)$, $t \geq 1$, and private signal $s_t \in S$.*

Proof. Let's fix some t, history h_t, and signal s_t. Suppose, for contradiction, $\zeta_t(h_t^x, s_t) > 0$ and $\iota_t(h_t^x, s_t) > 0$, it implies, by optimality, that

$$\min_{\pi,p} E^{\pi\times p} [u(v - a_t)|h_t, s_t] = \min_{\pi,p} E^{\pi\times p} [u(b_t - v)|h_t, s_t] \geq 0,$$

and by Jensen's inequality,

$$\min_{\pi,p} E^{\pi\times p}\left[v - a_t | h_t, s_t\right] \geq u^{-1}\left[\min_{\pi,p} E^{\pi\times p}\left[u(v - a_t)|h_t, s_t\right]\right]$$
$$\geq u^{-1}(0) = 0,$$
$$\implies a_t \leq \min_{\pi,p} E^{\pi\times p}\left[v|h_t, s_t\right]$$

and similarly,

$$b_t \geq \max_{\pi,p} E^{\pi\times p}\left[v|h_t, s_t\right],$$

and hence,

$$b_t \geq \max_{\pi,p} E^{\pi\times p}\left[v|h_t, s_t\right] \geq \min_{\pi,p} E^{\pi\times p}\left[v|h_t, s_t\right] \geq a_t.$$

But, by Proposition 3.2.3, $a_t < b_t$, and hence a contradiction. □

Proof of the Theorem

Proof. See Proof of Proposition 3.2.3 and Lemma C.1.4. □

Proof of the Proposition 3.3.1.

Proof. First, we show the second part of the proposition regarding the prices and trading decisions in the equilibrium.

I claim that it cannot be the following cases at any history in equilibrium:

(i) $\hat{z}_t(h_t^x, 1) = \hat{z}_t(h_t^x, 0)$;

(ii) $\hat{z}_t(h_t^x, 1) < \hat{z}_t(h_t^x, 0)$;

(iii) $\hat{z}_t(h_t^x, 0) = 0$ or $\hat{z}_t(h_t^x, 1) = 0$.

Proof. (For the claim). For (i) there would be three cases:

$$(1)\ \hat{z}_t(h_t^x, 1) = \hat{z}_t(h_t^x, 0) = 1$$
$$(2)\ \hat{z}_t(h_t^x, 1) = \hat{z}_t(h_t^x, 0) = 0$$
$$(3)\ \hat{z}_t(h_t^x, 1) = \hat{z}_t(h_t^x, 0) = -1$$

For (1), suppose, for contradiction, that there exist such a history h_t, then the market maker's zero profit condition would be:

$$\frac{1-\eta}{3}\left(a_t(h_t) - \pi_t\right) + \eta\left(a_t(h_t) - \pi_t\right) = 0$$
$$\frac{1-\eta}{3}\left(\pi_t - b_t(h_t)\right) = 0,$$

which implies that $a_t(h_t) = b_t(h_t) = \pi_t$. Then,

$$\hat{z}_t(h_t^x, 0) = 1 \implies \pi_t^{(0)} \geq a_t(h_t) = \pi_t,$$

but

$$\pi_t^{(0)} = \frac{\pi_t(1-q)}{\pi_t(1-q)+(1-\pi_t)q} < \pi_t$$

given that $q > 1/2$, a contradiction. Proofs are similar for (2) and (3).

For (ii) there would be three cases:

$$\begin{aligned}
&(1)\ \hat{z}_t(h_t^x, 0) = 1 \text{ and } \hat{z}_t(h_t^x, 1) = 0\\
&(2)\ \hat{z}_t(h_t^x, 0) = 1 \text{ and } \hat{z}_t(h_t^x, 1) = -1\\
&(3)\ \hat{z}_t(h_t^x, 0) = 0 \text{ and } \hat{z}_t(h_t^x, 1) = -1.
\end{aligned}$$

For (2), suppose there exist such a history h_t, then the market maker's zero profit condition would be:

$$\frac{1-\eta}{3}\left(a_t(h_t)-\pi_t\right)+\eta\left[\pi_t(1-q)\left(a_t(h_t)-1\right)+(1-\pi_t)qa_t(h_t)\right]=0 \qquad \text{(C.1.1)}$$

$$\frac{1-\eta}{3}\left(\pi_t-b_t(h_t)\right)+\eta\left[\pi_t q\left(1-b_t(h_t)\right)+(1-\pi_t)(1-q)\left(-b_t(h_t)\right)\right]=0, \qquad \text{(C.1.2)}$$

which yields that

$$a_t(h_t) = \frac{\frac{1-\eta}{3\eta}\pi_t + \pi_t(1-q)}{\frac{1-\eta}{3\eta} + \pi_t(1-q)+(1-\pi_t)q} \in (\pi_t^{(0)}, \pi_t).$$

However,

$$\hat{z}_t(h_t^x, 0) = 1 \implies \pi_t^{(0)} \geq a_t(h_t),$$

a contradiction. Proofs are similar for (1) and (3).

For (iii), suppose, for contradiction, there exists such a history h_t that $\hat{z}_t(h_t^x, 0) = 0$. Then, by (i) and (ii), it implies that $\hat{z}_t(h_t^x, 1) = 1$.

Then, the market maker's conditions are

$$\frac{1-\eta}{3}\left(\pi_t - b_t(h_t)\right) = 0,$$

which yields that

$$b_t(h_t) = \pi_t.$$

And note that,

$$\hat{z}_t(h_t^x, 0) = 0 \implies b_t(h_t) \leq \pi_t^{(0)} \leq a_t(h_t);$$

but,

$$\pi_t^{(0)} = \frac{\pi_t(1-q)}{\pi_t(1-q)+(1-\pi_t)q} < \pi_t = b_t(h_t),$$

given that $q > 1/2$, a contradiction. The argument for non-possibility of $\hat{z}_t(h_t^x, 1) = 0$ will be similar. $\square$

Therefore, for any history h_t in equilibrium, we will have $\hat{z}_t(h_t^x, 0) = -1$ and $\hat{z}_t(h_t^x, 1) = 1$. Then, in this case, the market maker's zero profit conditions are

$$\frac{1-\eta}{3}\left(a_t(h_t) - \pi_t\right) + \eta\left[\pi_t q\left(a_t(h_t) - 1\right) + (1-\pi_t)(1-q)a_t(h_t)\right] = 0$$
$$\frac{1-\eta}{3}\left(\pi_t - b_t(h_t)\right) + \eta\left[\pi_t(1-q)\left(1 - b_t(h_t)\right) - (1-\pi_t)q b_t(h_t)\right] = 0,$$

which yields the unique solution that

$$a_t(h_t) = \frac{\frac{1-\eta}{3\eta}\pi_t + \pi_t q}{\frac{1-\eta}{3\eta} + \pi_t q + (1-\pi_t)(1-q)} \in (\pi_t, \pi_t^{(1)}),$$
$$b_t(h_t) = \frac{\frac{1-\eta}{3\eta}\pi_t + \pi_t(1-q)}{\frac{1-\eta}{3\eta} + \pi_t(1-q) + (1-\pi_t)q} \in (\pi_t^{(0)}, \pi_t),$$

apparently, then we have $\pi_t^{(0)} < b_t(h_t) < \pi_t < a_t(h_t) < \pi_t^{(1)}$ as desired.

For the existence of equilibrium, first, the trading strategy is comparing the updated belief with the ask and bid prices, and pricing strategies as specified by equations (3.3.1) and (3.3.2) in each period and given any public belief. We can verify that, given such $a_t(\cdot)$ and $b_t(\cdot)$, $z_t(h_t^x, 1) = 1$ and $z_t(h_t^x, 0) = -1$ as prescribed by the trading strategies. Therefore, $a_t(h_t)$ and $b_t(h_t)$ need to satisfy the zero profit conditions (C.1.1) and (C.1.2), and it is true since equations (3.3.1) and (3.3.2) are equivalent to zero profit conditions. And as long as we have beliefs to be consistent (w.r.t. the specified strategies), the existence of the equilibrium will then be established.

To look at the belief updating, under the above specified strategies, and at the end of period 1, given π_1, and a_1, b_1 determined by the strategy, and z_1 records the trading order at period 1, then

$$\begin{aligned}
\pi_2(h_2) &= \Pr(v = 1 | h_2) \\
&= \frac{\Pr(v = 1, h_2)}{\Pr(h_1)} \\
&= \frac{\Pr(v = 1, z_1)}{\Pr(z_1)} \\
&= \frac{\Pr(z_1 | v = 1)\pi_1}{\Pr(z_1 | v = 1)\pi_1 + \Pr(z_1 | v = 0)(1 - \pi_1)}
\end{aligned}$$

At the end of period $t > 1$, given h_t,$\pi_t(h_t)$, and a_t, b_t, z_t,

$$\pi_{t+1}(h_{t+1}) = \frac{\Pr(z_t | h_t, v = 1)\pi_t(h_t)}{\Pr(z_t | h_t, v = 1)\pi_t(h_t) + \Pr(z_t | h_t, v = 0)\left[1 - \pi_t(h_t)\right]}$$

Note that at any history h_t, the conditional probabilities of a "buy" is

$$\Pr(z_t = 1|h_t, v = 1) = \frac{1-\eta}{3} + \eta q$$
$$\Pr(z_t = 1|h_t, v = 0) = \frac{1-\eta}{3} + \eta(1-q),$$

and conditional probability of a "sell" is

$$\Pr(z_t = -1|h_t, v = 1) = \frac{1-\eta}{3} + \eta(1-q)$$
$$\Pr(z_t = -1|h_t, v = 0) = \frac{1-\eta}{3} + \eta q.$$

The last step is simply to verify that the specified strategy $a_t(\cdot), b_t(\cdot), \hat{z}_t$ and belief system $\{\pi_t\}$ satisfy the conditions to be an equilibrium, which is omitted. □

Proof of Lemma 3.3.2

Proof. To prove the lemma, we will show that none of the following is possible for any history h_t in equilibrium:

(i) $\hat{z}_t(h_t^x, 1) = \hat{z}_t(h_t^x, 0) = 1$ or $\hat{z}_t(h_t^x, 1) = \hat{z}_t(h_t^x, 0) = -1$;
(ii) $\hat{z}_t(h_t^x, 1) < \hat{z}_t(h_t^x, 0)$;
(iii) $\hat{z}_t(h_t^x, 1) = 1$ and $\hat{z}_t(h_t^x, 0) = 0$; or
(iv) $\hat{z}_t(h_t^x, 1) = 0$ and $\hat{z}_t(h_t^x, 0) = -1$.

The idea of proving non-possibility of these four cases are similar. We compute the ask and bid price under each case, then conclude it with some contradiction.

For case (i), from market maker's conditions

$$\hat{z}_t(h_t^x, 1) = \hat{z}_t(h_t^x, 0) = 1 \implies a_t(h_t) = \bar{\pi}_t \text{ and } b_t(h_t) = \underline{\pi}_t,$$

but, from the informed agent's decision,

$$\hat{z}_t(h_t^x, 0) = 1 \implies a_t(h_t) < \underline{\pi}_t^{(0)} < \underline{\pi}_t,$$

a contradiction. Proof for the case of $\hat{z}_t(h_t^x, 1) = \hat{z}_t(h_t^x, 0) = -1$ is similar.

For case (ii), There would several cases: $\hat{z}_t(h_t^x, 0) = 1$ and $\hat{z}_t(h_t^x, 1) = 0$; $\hat{z}_t(h_t^x, 0) = 1$ and $\hat{z}_t(h_t^x, 1) = -1$; $\hat{z}_t(h_t^x, 0) = 0$ and $\hat{z}_t(h_t^x, 1) = -1$. The proofs would be similar, and we will only prove the case $\hat{z}_t(h_t^x, 0) = 1$ and $\hat{z}_t(h_t^x, 1) = 0$.

From market maker's conditions

$$\hat{z}_t(h_t^x, 0) = 1 \implies a_t(h_t) = \frac{\gamma\bar{\pi}_t + \bar{\pi}_t(1-q)}{\gamma + \bar{\pi}_t(1-q) + (1-\bar{\pi}_t)q} \in (\bar{\pi}_t^{(0)}, \bar{\pi}_t),$$

but from the informed agent's decision,

$$\hat{z}_t(h_t^x, 0) = 1 \implies a_t(h_t) \leq \underline{\pi}_t^{(0)};$$

therefore,

$$a_t(h_t) \le \underline{\pi}_t^{(0)} \le \bar{\pi}_t^{(0)} < a_t(h_t),$$

a contradiction.

For case (iii) and (iv), they would be similar, and we only show (iii). In fact, we can show that,

$$\hat{z}_t(h_t^x, 1) = 1 \implies \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} \le \frac{q}{1-q}\frac{1-q+\gamma}{q+\gamma} < \frac{q}{1-q}$$

since $\frac{1-q+\gamma}{q+\gamma} \in \left[\frac{1-q}{q}, 1\right)$, and

$$\hat{z}_t(h_t^x, 0) = 0 \implies \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} \ge \frac{q}{1-q},$$

a contradiction. □

Proof of Proposition 3.3.5

Proof. Note that (ii) and (iii) is just Proposition 3.3.2 and 3.3.3. So the proof focuses on (i). Fix an equilibrium and a history h_t in equilibrium, let $\Pi_t(h_t)$ be given as well.

(Only if side). Given some equilibrium and history h_t in equilibrium and $\zeta_t(h_t^x, 1), \iota_t(h_t^x, 0) \in (0, 1)$, the market maker's conditions can be written explicitly as

$$\min_{\pi_t \in \Pi_t(h_t)} \frac{1-\eta}{3}\left(a_t(h_t) - \pi_t\right) + \eta\zeta_t\left[\pi_t q\left(a_t(h_t) - 1\right) + (1-\pi_t)(1-q)a_t(h_t)\right] = 0$$
$$\min_{\pi_t \in \Pi_t(h_t)} \frac{1-\eta}{3}\left(\pi_t - b_t(h_t)\right) + \eta\iota_t\left[\pi_t(1-q)\left(1 - b_t(h_t)\right) - (1-\pi_t)q b_t(h_t)\right] = 0,$$

where ζ_t is short for $\zeta_t(h_t^x, 1)$ and ι_t is short for $\iota_t(h_t^x, 0)$, and it yields that

$$a_t(h_t) = \frac{\bar{\pi}_t\gamma/\zeta_t + \bar{\pi}_t q}{\gamma/\zeta_t + \bar{\pi}_t q + (1-\bar{\pi}_t)(1-q)}$$
$$b_t(h_t) = \frac{\underline{\pi}_t\gamma/\iota_t + \underline{\pi}_t(1-q)}{\gamma/\iota_t + \underline{\pi}_t(1-q) - (1-\underline{\pi}_t)q}.$$

Moreover, from the trading strategy, see Proposition **??**, $\zeta_t(h_t^x, 1), \iota_t(h_t^x, 0) \in (0, 1)$ implies that

$$a_t(h_t) = \underline{\pi}_t^{(1)}$$
$$b_t(h_t) = \bar{\pi}_t^{(0)}.$$

Therefore, rearranging them, we can get

$$\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} = \frac{q}{1-q}\frac{1-q+\gamma/\zeta_t}{q+\gamma/\zeta_t}$$
$$\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} = \frac{q}{1-q}\frac{1-q+\gamma/\iota_t}{q+\gamma/\iota_t}.$$

First we have

$$\frac{1-q+\gamma/\zeta_t}{q+\gamma/\zeta_t} = \frac{1-q+\gamma/\iota_t}{q+\gamma/\iota_t},$$

but since $\frac{1-q+\gamma}{q+\gamma}$ is strictly increasing in γ, it implies that $\zeta_t = \iota_t = \sigma$ for some σ. That is as an implication of the symmetry in the assumptions, the probability of buying with a good signal and probability of selling with a bad signal are the same.

Moreover, notice that

$$\lim_{\sigma\to 0}\frac{1-q+\gamma/\sigma}{q+\gamma/\sigma} = 1$$
$$\lim_{\sigma\to 1}\frac{1-q+\gamma/\sigma}{q+\gamma/\sigma} = \frac{1-q+\gamma}{q+\gamma}$$

and $\frac{1-q+\gamma/\sigma}{q+\gamma/\sigma}$ is strictly decreasing in σ, so for any $r(h_t)$ defined by

$$r(h_t) = \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}\frac{1-q}{q},$$

there exist a unique $\sigma(h_t)$ such that it solves

$$r(h_t) = \frac{1-q+\gamma/\sigma}{q+\gamma/\sigma},$$

if $r(h_t) \in \left(\frac{1-q+\gamma}{q+\gamma}, 1\right)$, which is equivalent to (3.3.13). Therefore, this tells us that if the informed traders are playing the strict mixed strategies, then (3.3.13) must hold.

(If side). We will only show that $a_t(h_t) = \underline{\pi}_t^{(1)}$ since the proof of $b_t(h_t) = \bar{\pi}_t^{(0)}$ is similar.

Suppose, for contradiction, $a_t(h_t) > \underline{\pi}_t^{(1)}$. Then, from the trading strategy, $\zeta_t(h_t^x, 1) = 0$. However, then, by market maker's condition, $a_t(h_t) = \bar{\pi}_t$. And $\bar{\pi}_t > \underline{\pi}_t^{(1)} \implies \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} > \frac{q}{1-q}$, a contradiction.

Suppose, for contradiction, $a_t(h_t) < \underline{\pi}_t^{(1)}$. Then, from the trading strategy, $\zeta_t(h_t^x, 1) = 1$. Then, by market maker's condition,

$$a_t(h_t) = \frac{\bar{\pi}_t\gamma + \bar{\pi}_t q}{\gamma + \bar{\pi}_t q + (1-\bar{\pi}_t)(1-q)},$$

and

$$\frac{\bar{\pi}_t\gamma+\bar{\pi}_t q}{\gamma+\bar{\pi}_t q+(1-\bar{\pi}_t)(1-q)}<\underline{\pi}_t^{(1)}$$
$$\Longrightarrow \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}<\frac{q}{1-q}\frac{1-q+\gamma}{q+\gamma},$$

a contradiction. □

Proof of Proposition 3.3.7

Proof. The only if sides in each case are implications of Proposition 3.3.5 and 3.3.6. We show the if side of case (i) and the proofs for others are similar.

Let inequality (3.3.8) hold. If we set

$$a_t(h_t)<\frac{\bar{\pi}_t\gamma+\bar{\pi}_t q}{\gamma+\bar{\pi}_t q+(1-\bar{\pi}_t)(1-q)},$$

then the informed agent's decision is not changing and the min-expected profit of a "buy", i.e.

$$\min_{p\in\mathcal{P},\pi\in\Pi_t(h_t)} E^{\pi\times p}\left[\left(\frac{1-\eta}{3}+\eta\zeta_t(h_t,a_t,b_t,s_t)\right)(a_t-v)\,|h_t\right]$$

is then strictly increasing in a_t, hence it must be strictly less than 0, contradictory to the zero min-expected profit condition. Hence, it must be such that

$$a_t(h_t)\geq\frac{\bar{\pi}_t\gamma+\bar{\pi}_t q}{\gamma+\bar{\pi}_t q+(1-\bar{\pi}_t)(1-q)}.$$

Similar argument applies so that it must be the case that

$$b_t\leq\frac{\underline{\pi}_t\gamma+\underline{\pi}_t(1-q)}{\gamma+\underline{\pi}_t(1-q)-(1-\underline{\pi}_t)q}.$$

But if we set

$$a_t(h_t)>\frac{\bar{\pi}_t\gamma+\bar{\pi}_t q}{\gamma+\bar{\pi}_t q+(1-\bar{\pi}_t)(1-q)}>\bar{\pi}_t,$$

then there will be two cases sine inequality (3.3.8) implies that $\underline{\pi}_t^{(1)}>\bar{\pi}_t$. One is that $a_t(h_t)\leq\underline{\pi}_t^{(1)}$, again, the informed agent's decision is not changing and the min-expected profit of a "buy" is then strictly increasing in a_t, hence it must be strictly greater than 0. The other case is that $a_t(h_t)>\underline{\pi}_t^{(1)}$, then no informed traders will buy even with a good signal, however, then the min-expected profit of a "buy" is $\frac{1-\eta}{3}\left[a_t(h_t)-\bar{\pi}_t\right]>0$. So both cases are contradictory to the zero min-expected profit condition. And apply the similar argument, we shall see the impossibility of

$$b_t<\frac{\underline{\pi}_t\gamma+\underline{\pi}_t(1-q)}{\gamma+\underline{\pi}_t(1-q)-(1-\underline{\pi}_t)q}.$$

Therefore, $a_t(h_t)$ and $b_t(h_t)$ have to be that as stated in the proposition. □

Proof of Theorem 3.3.1

Proof. Let $\Pi_t'(h_t) = [\underline{\pi}_t', \bar{\pi}_t']$ and $\Pi_t(h_t) = [\underline{\pi}_t, \bar{\pi}_t]$ be given and $\Pi_t'(h_t)$ is more ambiguous than $\Pi_t(h_t)$. Note that, in equilibrium, $\zeta_t(h_t^x, 1) = \iota_t(h_t^x, 0) = \sigma$. $\Pi_t'(h_t)$ is more ambiguous than $\Pi_t(h_t)$ implies that $\frac{\bar{\pi}_t'(1-\underline{\pi}_t')}{\underline{\pi}_t'(1-\bar{\pi}_t')} \geq \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}$.

Then, I claim that σ is weakly decreasing in $\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}$. Denote $\zeta_t'(h_t^x, 1) = \iota_t'(h_t^x, 0) = \sigma'$. Then, there can be several cases: (1) $\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} \geq \frac{q}{1-q}$, 2) $\frac{\bar{\pi}_t'(1-\underline{\pi}_t')}{\underline{\pi}_t'(1-\bar{\pi}_t')} \leq \frac{q}{1-q}\frac{1-q+\gamma}{q+\gamma}$, 3) $\frac{\bar{\pi}_t'(1-\underline{\pi}_t')}{\underline{\pi}_t'(1-\bar{\pi}_t')} \geq \frac{q}{1-q} \geq \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}$, 4) $\frac{\bar{\pi}_t'(1-\underline{\pi}_t')}{\underline{\pi}_t'(1-\bar{\pi}_t')} \geq \frac{q}{1-q}\frac{1-q+\gamma}{q+\gamma} \geq \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}$, and 5) $\frac{q}{1-q} \geq \frac{\bar{\pi}_t'(1-\underline{\pi}_t')}{\underline{\pi}_t'(1-\bar{\pi}_t')} \geq \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} \geq \frac{q}{1-q}\frac{1-q+\gamma}{q+\gamma}$. It is trivial to show $\sigma' \leq \sigma$ in the first four cases. So let's focus on the last case. From Proof of Proposition 3.3.5, in that case,

$$\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} = \frac{q}{1-q}\frac{1-q+\gamma/\sigma}{q+\gamma/\sigma} \text{ and } \frac{\bar{\pi}_t'(1-\underline{\pi}_t')}{\underline{\pi}_t'(1-\bar{\pi}_t')} = \frac{q}{1-q}\frac{1-q+\gamma/\sigma'}{q+\gamma/\sigma'}$$

and $\frac{1-q+\gamma/\sigma}{q+\gamma/\sigma}$ is strictly decreasing in σ. Therefore, $\sigma' \leq \sigma$.

For the ask prices, note that since $\frac{\pi(\gamma+q\sigma)}{\gamma+[\pi q+(1-\pi)(1-q)]\sigma}$ is increasing in π and increasing in σ,

$$\begin{aligned} a_t'(h_t) &= \frac{\bar{\pi}_t'(\gamma+q\sigma')}{\gamma+[\bar{\pi}_t' q+(1-\bar{\pi}_t')(1-q)]\,\sigma'} \\ &\geq \frac{\bar{\pi}_t(\gamma+q\sigma)}{\gamma+[\bar{\pi}_t q+(1-\bar{\pi}_t)(1-q)]\,\sigma} = a_t(h_t), \end{aligned}$$

where the inequality holds due to that $\bar{\pi}_t' \geq \bar{\pi}_t$ and $\sigma' \leq \sigma$ and the inequality is strict if $\bar{\pi}_t' > \bar{\pi}_t$. Similarly, $b_t'(h_t) \leq b_t(h_t)$ with strict inequality if $\underline{\pi}_t' < \underline{\pi}_t$. □

Proof of Proposition 3.4.2

Proof. First, we can show that

$$\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} < \frac{q}{1-\underline{q}}$$

for all $t = 1, 2, \ldots$ by mathematical induction. We know that it is true when $t = 1$ by assumption, and suppose that it is true for some $t < \infty$, then there could be two cases, either

$$\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} \leq \frac{\underline{q}}{1-\underline{q}}\frac{1-\bar{q}+\gamma}{\bar{q}+\gamma}$$

or

$$\frac{\underline{q}}{1-\underline{q}}\frac{1-\bar{q}+\gamma}{\bar{q}+\gamma} < \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} < \frac{\underline{q}}{1-\underline{q}}.$$

If

$$\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} \leq \frac{\underline{q}}{1-\underline{q}}\frac{1-\bar{q}+\gamma}{\bar{q}+\gamma},$$

then

$$\frac{l_t(1;\bar{q})}{l_t(1;\underline{q})} = \frac{\bar{q}+\gamma}{1-\bar{q}+\gamma}\frac{1-\underline{q}+\gamma}{\underline{q}+\gamma};$$

therefore,

$$\begin{aligned}\frac{\bar{\pi}_{t+1}(1-\underline{\pi}_{t+1})}{\underline{\pi}_{t+1}(1-\bar{\pi}_{t+1})} &\leq \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}\frac{l(h_t,1;\bar{q})}{l(h_t,1;\underline{q})} \\ &\leq \frac{\underline{q}}{1-\underline{q}}\frac{1-\underline{q}+\gamma}{\underline{q}+\gamma} \\ &< \frac{\underline{q}}{1-\underline{q}}.\end{aligned}$$

If

$$\frac{\underline{q}}{1-\underline{q}}\frac{1-\bar{q}+\gamma}{\bar{q}+\gamma} < \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} < \frac{\underline{q}}{1-\underline{q}},$$

then there exists $\sigma \in (0,1)$ such that

$$\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} = \frac{\underline{q}}{1-\underline{q}}\frac{1-\bar{q}+\gamma/\sigma}{\bar{q}+\gamma/\sigma}$$

and

$$\frac{l(h_t,1;\bar{q})}{l(h_t,1;\underline{q})} = \frac{\bar{q}+\gamma/\sigma}{1-\bar{q}+\gamma/\sigma}\frac{1-\underline{q}+\gamma/\sigma}{\underline{q}+\gamma/\sigma};$$

therefore

$$\begin{aligned}\frac{\bar{\pi}_{t+1}(1-\underline{\pi}_{t+1})}{\underline{\pi}_{t+1}(1-\bar{\pi}_{t+1})} &\leq \frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)}\frac{l(h_t,1;\bar{q})}{l(h_t,1;\underline{q})} \\ &= \frac{\underline{q}}{1-\underline{q}}\frac{1-\underline{q}+\gamma/\sigma}{\underline{q}+\gamma/\sigma} \\ &< \frac{\underline{q}}{1-\underline{q}}.\end{aligned}$$

Fix some $\varepsilon > 0$ and first I claim that there exists some $T_\varepsilon < \infty$ such that $\frac{\bar{\pi}_{T_\varepsilon}(1-\underline{\pi}_{T_\varepsilon})}{\underline{\pi}_{T_\varepsilon}(1-\bar{\pi}_{T_\varepsilon})} > \frac{\underline{q}}{1-\underline{q}} - \varepsilon$ if $z_t \neq 0$ for all $t = 1, \ldots, T_\varepsilon$.

Then, we can show that for $t \geq T_\varepsilon$ large enough, say $\bar{t}$

$$\begin{aligned}
\Pr\left(\left|\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} - \frac{\underline{q}}{1-\underline{q}}\right| > \varepsilon\right) &= \Pr\left(\#\left\{z_\tau \neq 0, \tau \leq t\right\} < T_\varepsilon\right) \\
&= \sum_{k<T_\varepsilon} \left(\begin{array}{c} t \\ k \end{array}\right) \left(\frac{2(1-\eta)}{3} + \eta\sigma\right)^k \left(\frac{1-\eta}{3} + \eta(1-\sigma)\right)^{t-k} \\
&\leq \sum_{k<T_\varepsilon} \left(\begin{array}{c} t \\ k \end{array}\right) /2^t.
\end{aligned}$$

Note that $\sum_{t \geq T_\varepsilon} \sum_{k<T_\varepsilon} \left(\begin{array}{c} t \\ k \end{array}\right) /2^t < \infty$, so we have

$$\sum_{t=1}^{\infty} \Pr\left(\left|\frac{\bar{\pi}_t(1-\underline{\pi}_t)}{\underline{\pi}_t(1-\bar{\pi}_t)} - \frac{\underline{q}}{1-\underline{q}}\right| > \varepsilon\right) < \infty,$$

so we can conclude the almost sure convergence.

$\square$